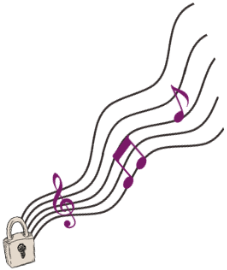
Cryptanalysis and design of symmetric primitives defined over large finite fields



Clémence Bouvier

November 27th, 2023

SORBONNE UNIVERSITÉ

Inria

Toy example of Zero-Knowledge Proof

	2		5		1		9	
8			2		3			6
	3			6			7	
		1				6		
5	4						1	9
		2				7		
	9			3			8	
2			8		4			7
	1		9		7		6	

Unsolved Sudoku

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2			8		4			7
	1		9		7		6	

Unsolved Sudoku

→

4	2	*6*	5	*7*	1	*3*	9	*8*
8	*5*	*7*	2	*9*	3	*1*	*4*	6
1	3	*9*	*4*	6	*8*	*2*	7	*5*
9	*7*	1	*3*	*8*	*5*	6	*2*	*4*
5	4	*3*	*7*	*2*	*6*	*8*	1	9
6	*8*	2	*1*	*4*	*9*	7	*5*	*3*
7	9	*4*	*6*	3	*2*	*5*	8	*1*
2	*6*	*5*	8	*1*	4	*9*	*3*	7
3	1	*8*	9	*5*	7	*4*	6	*2*

Solved Sudoku

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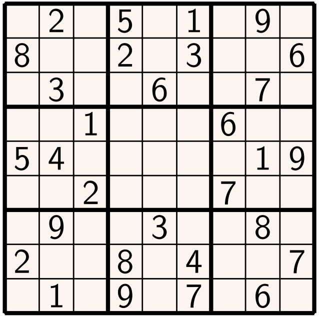
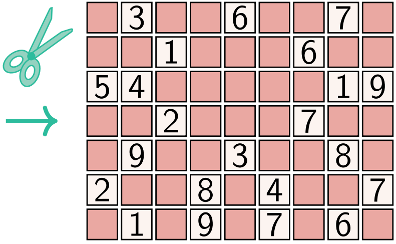


Unsolved Sudoku



Grid cutting

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	2		5		1		9	
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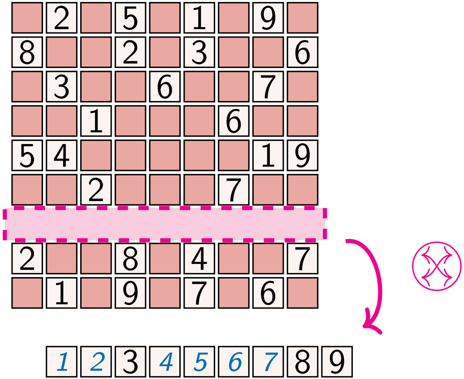

Rows checking

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	2		5		1		9	
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	3			6			7	
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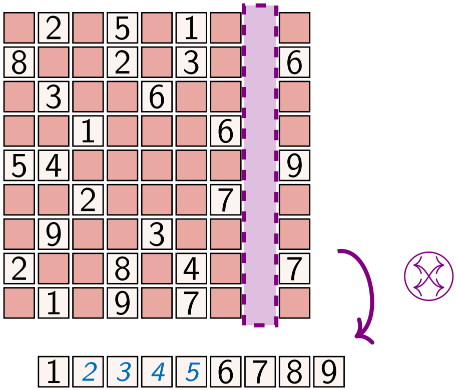
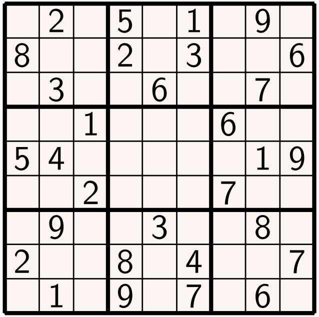


Columns checking

Toy example of Zero-Knowledge Proof



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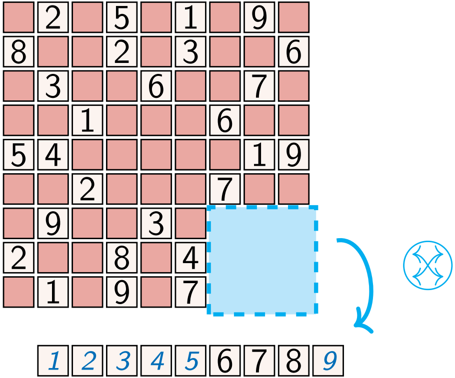
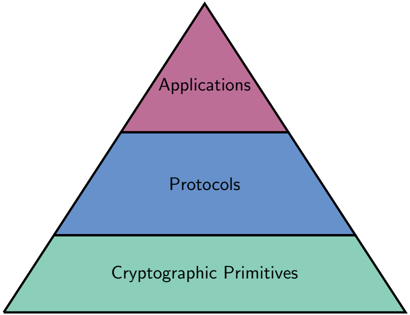


Squares checking

A NEW CONTEXT

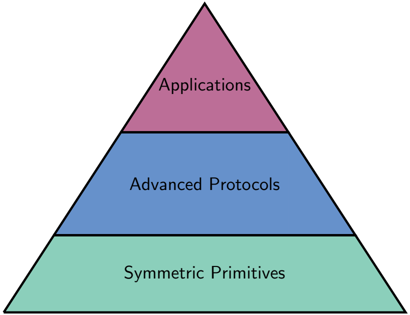
A need for new primitives



A need for new primitives

Protocols requiring new primitives:

- ⋆ **MPC**: Multiparty Computation
- ⋆ **FHE**: Fully Homomorphic Encryption
- ⋆ **ZK**: Systems of Zero-Knowledge proofs
  Example: SNARKs, STARKs, Bulletproofs



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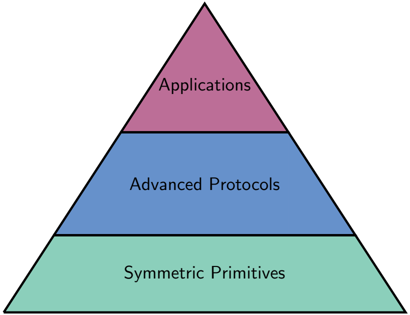


Problem: Designing new symmetric primitives

And analyse their security!

Block ciphers

- ⋆ input: n-bit block

$$x \in \mathbb{F}_2^n$$

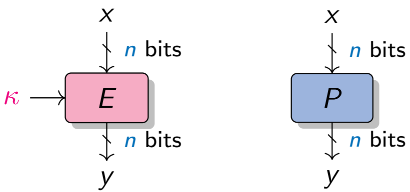
- ⋆ parameter: k-bit key

$$\kappa \in \mathbb{F}_2^k$$

- ⋆ output: n-bit block

$$y = E_\kappa(x) \in \mathbb{F}_2^n$$

- ⋆ symmetry: E and E^{-1} use the same κ



(a) *Block cipher* **(b)** *Random permutation*

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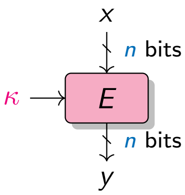
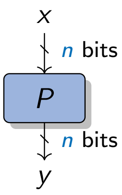
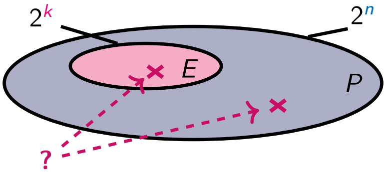


(a) *Block cipher*



(b) *Random permutation*

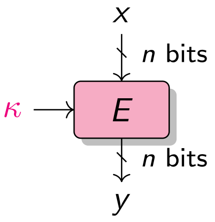


A block cipher is a family of 2^k permutations of $\mathbb{F}_2^n$.

Iterated constructions

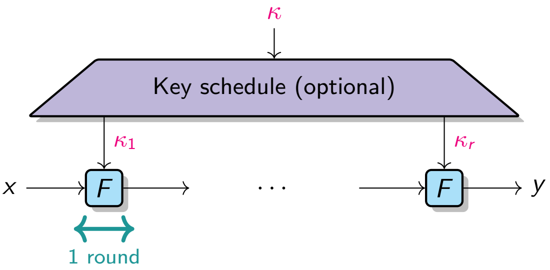
How to build an efficient block cipher?

By iterating a round function.



Block cipher

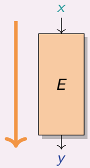
$\Rightarrow$



Comparison with the traditional case

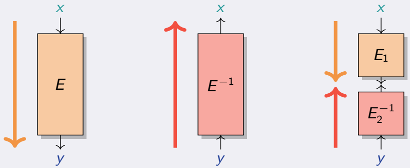
Traditional case

$$y \leftarrow E(x)$$



Arithmetization-oriented

$$y \leftarrow E(x) \quad \text{and} \quad y == E(x)$$



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- Alphabet size:
  $\mathbb{F}_2^n$, with $n \simeq 4, 8$

  Ex: Field of AES: $\mathbb{F}_{2^n}$ where $n = 8$

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$$y \leftarrow E(x) \quad \text{and} \quad y == E(x)$$

- Optimized for:
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- Alphabet size:
  $\mathbb{F}_q$, with $q \in \{2^n, p\}, p \simeq 2^n$, $n \geq 64$

  Ex: Scalar Field of Curve BLS12-381: $\mathbb{F}_p$ where

  $p =$ 0x73eda753299d7d483339d80809a1d805
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- Optimized for: implem… e/hardware
- … size: … $_2$, with $n \simeq 4, 8$
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Decades of Cryptanalysis

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$\leq$ 5 years of Cryptanalysis

Overview of the contributions

Design

Introducing the link between CCZ-equivalence and Arithmetization-Orientation → Designing a new S-Box: the `Flystel`

→ Designing a new family of ZK-friendly hash functions: `Anemoi`

Designing a new compression mode for Merkle trees: `Jive` → Designing a new family of ZK-friendly hash functions: `Anemoi`

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Cryptanalysis

Study of algebraic attacks against some Arithmetization-Oriented primitives

Study of the univariate polynomial representation of MiMC → Finding bounds on the algebraic degree of MiMC

Tracing maximum-weight exponents to reach the upper bound on the algebraic degree of MiMC

Investigating various open problems to better understand the behaviour of the algebraic degree

Overview of the contributions

Design

New Design Techniques for Efficient Arithmetization-Oriented Hash Functions: Anemoi Permutations and Jive Compression Mode, Bouvier, Briaud, Chaidos, Perrin, Salen, Velichkov, Willems, **CRYPTO 2023**

Cryptanalysis

Algebraic attacks against some arithmetization-oriented primitives, Bariant, Bouvier, Leurent, Perrin, **ToSC, 2022**

On the algebraic degree of iterated power functions, Bouvier, Canteaut, Perrin, **DCC, 2023**

Coefficient Grouping for Complex Affine Layers, Lui, Grassi, Bouvier, Meier, Isobe, **CRYPTO 2023**

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A new family of ZK-friendly hash functions: Anemoi

Performance metric

What does "efficient" mean for Zero-Knowledge Proofs?

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"It depends"

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Example

R1CS (Rank-1 Constraint System): minimizing the number of multiplications

$$y = (ax + b)^3 (cx + d) + ex$$

$t_0 = a \cdot x$	$t_3 = t_2 \times t_1$	$t_6 = t_3 \times t_5$
$t_1 = t_0 + b$	$t_4 = c \cdot x$	$t_7 = e \cdot x$
$t_2 = t_1 \times t_1$	$t_5 = t_4 + d$	$t_8 = t_6 + t_7$

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3 constraints

Our approach

Need: verification using few multiplications.

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⋆ **First approach:** evaluation using few multiplications, e.g. Poseidon [Grassi et al., USENIX21]

$y \leftarrow E(x)$ $\rightsquigarrow$ E: low degree

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⋆ **Our approach:** using $(u, v) = \mathcal{L}(x, y)$, where $\mathcal{L}$ is linear

$y \leftarrow F(x)$ $\rightsquigarrow$ F: high degree

$v == G(u)$ $\rightsquigarrow$ G: low degree

CCZ-equivalence

Inversion

$$\Gamma_F = \{(x, F(x)), x \in \mathbb{F}_q\} \quad \text{and} \quad \Gamma_{F^{-1}} = \left\{\left(y, F^{-1}(y)\right), y \in \mathbb{F}_q\right\}$$

Noting that

$$\Gamma_F = \left\{\left(F^{-1}(y), y\right), y \in \mathbb{F}_q\right\} \ ,$$

then, we have:

$$\Gamma_F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Gamma_{F^{-1}} \ .$$

CCZ-equivalence

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Definition [Carlet, Charpin and Zinoviev, DCC98]

$F : \mathbb{F}_q \to \mathbb{F}_q$ and $G : \mathbb{F}_q \to \mathbb{F}_q$ are **CCZ-equivalent** if

$$\Gamma_F = \mathcal{L}(\Gamma_G) + c \ , \quad \text{where } \mathcal{L} \text{ is linear.}$$

Advantages of CCZ-equivalence

If $F : \mathbb{F}_q \rightarrow \mathbb{F}_q$ and $G : \mathbb{F}_q \rightarrow \mathbb{F}_q$ are **CCZ-equivalent**. Then

⋆ Differential properties are the same: $\delta_F = \delta_G$.

Differential uniformity

Maximum value of the DDT

$$\delta_F = \max_{a \neq 0, b} |\{x \in \mathbb{F}_q^m, F(x+a) - F(x) = b\}|$$

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Linearity

Maximum value of the LAT

$$\mathcal{W}_F = \max_{a, b \neq 0} \left| \sum_{x \in \mathbb{F}_{2^n}^m} (-1)^{a \cdot x + b \cdot F(x)} \right|$$

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$$y == F(x)? \iff v == G(u)?$$

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- ⋆ The degree is **not preserved**.

Example

in $\mathbb{F}_p$ where

$$p = \texttt{0x73eda753299d7d483339d80809a1d80553bda402fffe5bfeffffffff00000001}$$

$$\text{if } F(x) = x^5 \text{ then } F^{-1}(x) = x^{5^{-1}} \text{ where}$$

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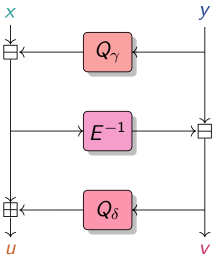
The Flystel

Butterfly + Feistel ⇒ Flystel

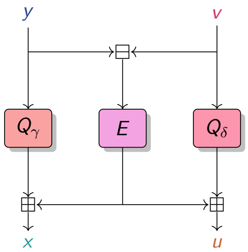
A 3-round Feistel-network with
$Q_\gamma : \mathbb{F}_q \to \mathbb{F}_q$ and $Q_\delta : \mathbb{F}_q \to \mathbb{F}_q$ two quadratic functions, and $E : \mathbb{F}_q \to \mathbb{F}_q$ a permutation

High-Degree permutation



Open Flystel $\mathcal{H}$.

Low-Degree function



Closed Flystel $\mathcal{V}$.

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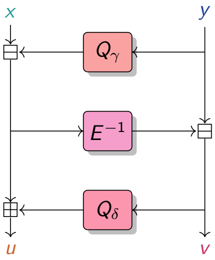

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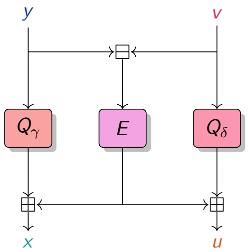


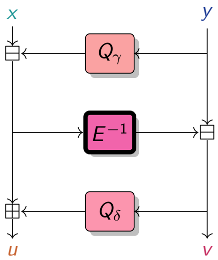
Closed Flystel $\mathcal{V}$.

$$\Gamma_{\mathcal{H}} = \mathcal{L}(\Gamma_{\mathcal{V}}) \quad \text{s.t.} \quad ((x,y),(u,v)) = \mathcal{L}\left(((v,y),(x,u)) \right)$$

Advantage of CCZ-equivalence

⋆ High-Degree Evaluation.

High-Degree permutation



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then $E^{-1} : x \mapsto x^{5^{-1}}$ where

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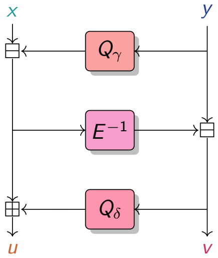
Advantage of CCZ-equivalence

- ⋆ High-Degree Evaluation.
- ⋆ Low-Degree Verification.

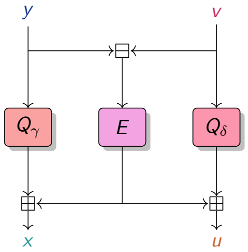
$$(u, v) == \mathcal{H}(x, y) \Leftrightarrow (x, u) == \mathcal{V}(y, v)$$

High-Degree permutation



Open `Flystel` $\mathcal{H}$.

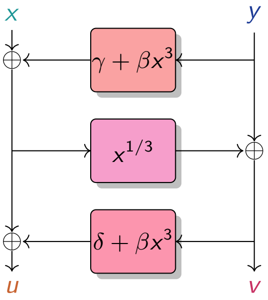
Low-Degree function



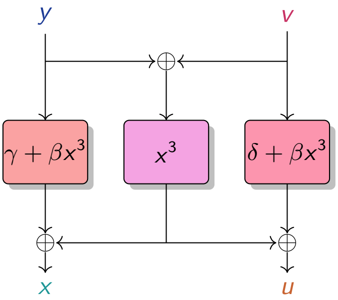
Closed `Flystel` $\mathcal{V}$.

Flystel in $\mathbb{F}_{2^n}$, n odd

$$Q_\gamma(x) = \gamma + \beta x^3 \ , \quad Q_\delta(x) = \delta + \beta x^3 \ , \quad \text{and} \quad E(x) = x^3$$

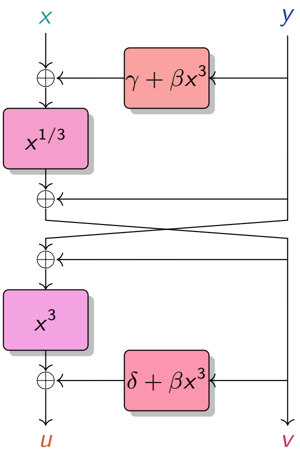


Open `Flystel`$_2$.



Closed `Flystel`$_2$.

Properties of Flystel in $\mathbb{F}_{2^n}$, n odd



Degenerated Butterfly.

Introduced by [Perrin et al. 2016].

Theorems in [Li et al. 2018] state that if $\beta \neq 0$:

- ⋆ Differential properties

$$\delta_{\mathcal{H}} = \delta_{\mathcal{V}} = 4$$

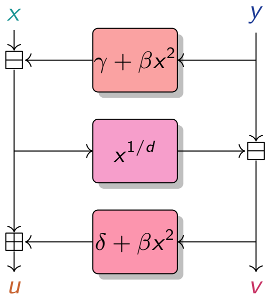
- ⋆ Linear properties

$$\mathcal{W}_{\mathcal{H}} = \mathcal{W}_{\mathcal{V}} = 2^{n+1}$$

- ⋆ Algebraic degree
  - ⋆ Open Flystel$_2$: $\deg_{\mathcal{H}} = n$
  - ⋆ Closed Flystel$_2$: $\deg_{\mathcal{V}} = 2$

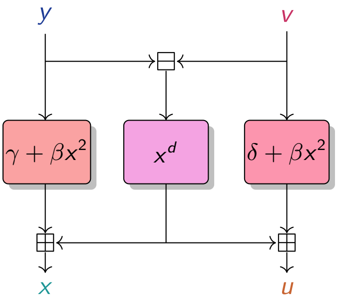
Flystel in $\mathbb{F}_p$

$$Q_\gamma(x) = \gamma + \beta x^2 \ , \quad Q_\delta(x) = \delta + \beta x^2 \ , \quad \text{and} \quad E(x) = x^d$$



Open Flystel$_p$.

usually
$d = 3$ or 5.



Closed Flystel$_p$.

Properties of Flystel in $\mathbb{F}_p$

⋆ Differential properties

$\texttt{Flystel}_\texttt{p}$ has a differential uniformity:

$$\delta_{\mathcal{H}} = \max_{a \neq 0, b} \left| \{ x \in \mathbb{F}_p^2, \mathcal{H}(x + a) - \mathcal{H}(x) = b \} \right| \leq d - 1$$

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Solving the open problem of finding an APN (Almost-Perfect Non-linear) permutation over $\mathbb{F}_p^2$

⋆ Linear properties

Conjecture:

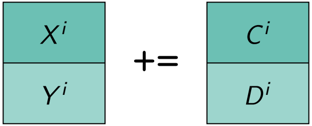
$$\mathcal{W}_{\mathcal{H}} = \max_{a, b \neq 0} \left| \sum_{x \in \mathbb{F}_p^2} exp\left(\frac{2\pi i(\langle a, x\rangle - \langle b, \mathcal{H}(x)\rangle)}{p}\right) \right| \leq p \log p \text{ ?}$$

The SPN Structure

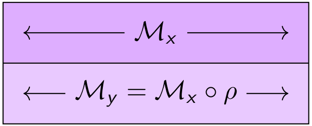
The internal state of Anemoi and its basic operations.

A Substitution-Permutation Network with:

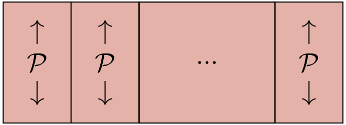
(a) *Internal state.*



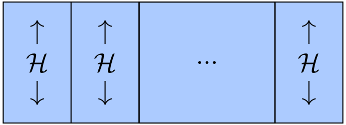
(b) *The constant addition.*



(c) *The diffusion layer.*

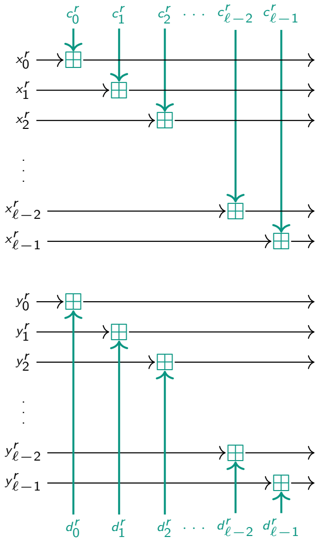


(d) *The Pseudo-Hadamard Transform.*



(e) *The S-box layer.*

The SPN Structure



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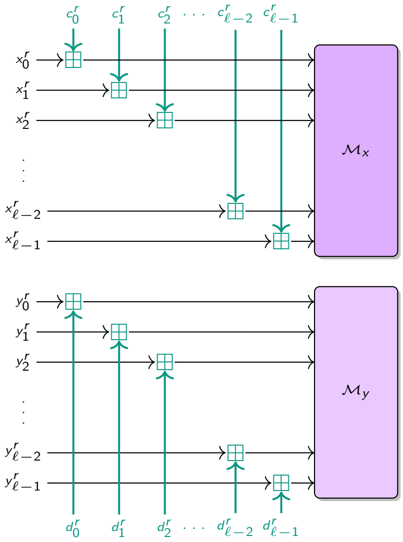

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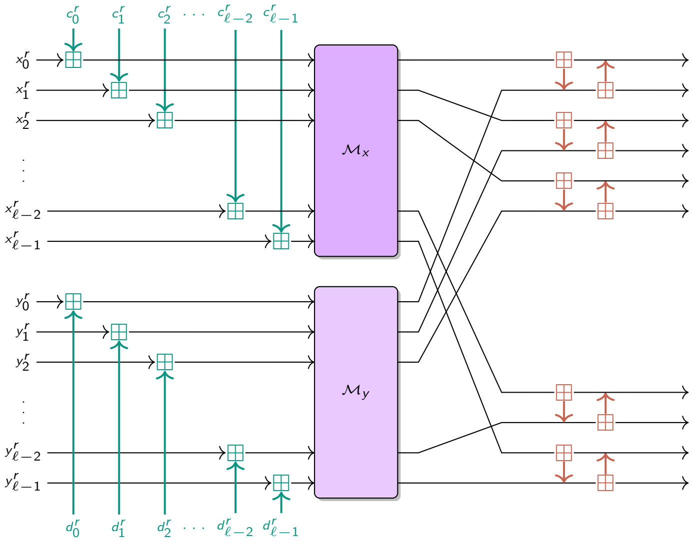

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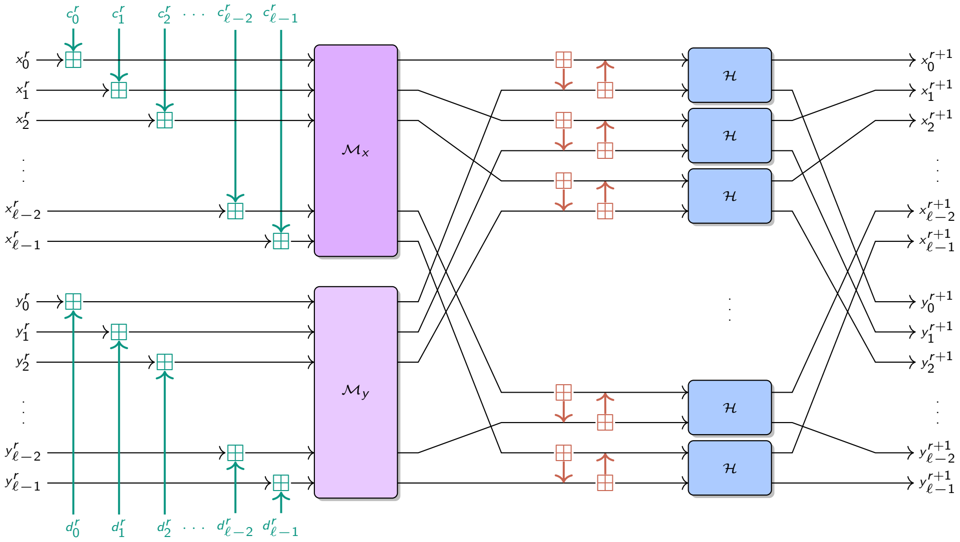

Number of rounds

$$\texttt{Anemoi}_{q,d,\ell} = \mathcal{M} \circ R_{n_r-1} \circ \ldots \circ R_0$$

⋆ Choosing the number of rounds

$$n_r \geq \max\left\{ 8\ ,\ \underbrace{\min(5, 1+\ell)}_{\text{security margin}} + \underbrace{2 + \min\left\{ r \in \mathbb{N} \;\middle|\; \binom{4\ell r + \kappa_d}{2\ell r}^2 \geq 2^s \right\}}_{\text{to prevent algebraic attacks}} \right\} .$$

d (κ_d)	3 (1)	5 (2)	7 (4)	11 (9)
$\ell = 1$	21	21	20	19
$\ell = 2$	14	14	13	13
$\ell = 3$	12	12	12	11
$\ell = 4$	12	12	11	11

Number of rounds of Anemoi ($s = 128$).

Performance metric

What does "efficient" mean for Zero-Knowledge Proofs?

"It depends"

Example

R1CS (Rank-1 Constraint System): minimizing the number of multiplications

$$y = (ax + b)^3 (cx + d) + ex$$

$$t_0 = a \cdot x \qquad t_3 = t_2 \times t_1 \qquad t_6 = t_3 \times t_5$$
$$t_1 = t_0 + b \qquad t_4 = c \cdot x \qquad t_7 = e \cdot x$$
$$t_2 = t_1 \times t_1 \qquad t_5 = t_4 + d \qquad t_8 = t_6 + t_7$$

3 constraints

Some Benchmarks

	$m\ (=2\ell)$	RP[1]	Poseidon[2]	Griffin[3]	Anemoi
R1CS	2	208	198	-	**76**
	4	224	232	112	**96**
	6	216	264	-	**120**
	8	256	296	176	**160**
Plonk	2	312	380	-	**191**
	4	560	832	**260**	316
	6	756	1344	-	**460**
	8	1152	1920	**574**	648
AIR	2	156	300	-	**126**
	4	**168**	348	**168**	**168**
	6	**162**	396	-	216
	8	**192**	456	264	288

(a) *when* $d = 3$.

	$m\ (=2\ell)$	RP	Poseidon	Griffin	Anemoi
R1CS	2	240	216	-	**95**
	4	264	264	**110**	120
	6	288	315	-	**150**
	8	384	363	**162**	200
Plonk	2	320	344	-	**212**
	4	528	696	**222**	344
	6	768	1125	-	**496**
	8	1280	1609	**492**	696
AIR	2	**200**	360	-	210
	4	**220**	440	**220**	280
	6	**240**	540	-	360
	8	**320**	640	360	480

(b) *when* $d = 5$.

Constraint comparison for standard arithmetization, without optimization ($s = 128$).

[1] *Rescue* [Aly et al., ToSC20]
[2] Poseidon [Grassi et al., USENIX21]
[3] Griffin [Grassi et al., CRYPTO23]

Take-Away

Anemoi: A new family of ZK-friendly hash functions

- ⋆ Identify a link between AO and CCZ-equivalence
- ⋆ Contributions of fundamental interest:
  - ⋆ New S-box: Flystel
  - ⋆ New mode: Jive

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Related works

- ⋆ $\texttt{AnemoiJive}_3$ with TurboPlonK [Liu et al., 2022]
- ⋆ Arion [Roy, Steiner and Trevisani, 2023]
- ⋆ APN permutations over prime fields [Budaghyan and Pal, 2023]

Cryptanalysis of MiMC

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Study of the corresponding sparse univariate polynomials

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Bounding the algebraic degree

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Tracing maximum-weight exponents reaching the upper bound

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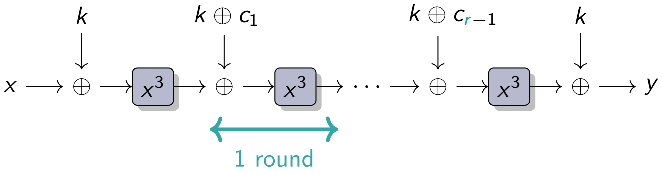
Study of higher-order differential attacks

The block cipher MiMC

- ⋆ Minimize the number of multiplications in $\mathbb{F}_{2^n}$.
- ⋆ Construction of MiMC$_3$ [Albrecht et al., AC16]:
  - ⋆ n-bit blocks (n odd ≈ 129): $x \in \mathbb{F}_{2^n}$
  - ⋆ n-bit key: $k \in \mathbb{F}_{2^n}$
  - ⋆ decryption : replacing x^3 by x^s where $s = (2^{n+1} - 1)/3$



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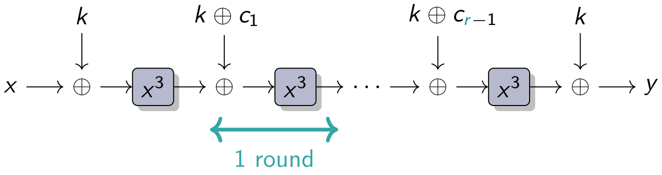
$$r := \lceil n \log_3 2 \rceil \ .$$

n	129	255	769	1025
r	82	161	486	647

Number of rounds for MiMC.



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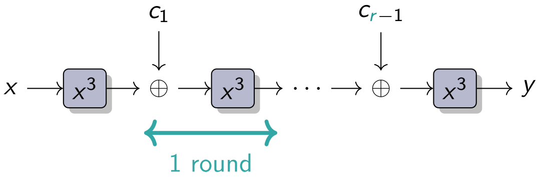

Algebraic degree - 1st definition

Let $f : \mathbb{F}_2^n \rightarrow \mathbb{F}_2$, there is **a unique multivariate polynomial** in $\mathbb{F}_2[x_1, \ldots x_n]/\left((x_i^2 + x_i)_{1 \le i \le n}\right)$:

$$f(x_1, ..., x_n) = \sum_{u \in \mathbb{F}_2^n} a_u x^u, \text{ where } a_u \in \mathbb{F}_2, \; x^u = \prod_{i=1}^{n} x_i^{u_i} \; .$$

This is the **Algebraic Normal Form (ANF)** of f.

Definition

Algebraic degree of $f : \mathbb{F}_2^n \rightarrow \mathbb{F}_2$:

$$\deg^a(f) = \max\left\{\text{wt}(u) : u \in \mathbb{F}_2^n, a_u \neq 0\right\} \; .$$

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If $F : \mathbb{F}_2^n \to \mathbb{F}_2^m$, with $F(x) = (f_1(x), \dots f_m(x))$, then

$$\deg^a(F) = \max\{\deg^a(f_i), \ 1 \le i \le m\} \ .$$

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This is the **Algebraic Normal Form (ANF)** of f.

Example: ANF of $x \mapsto x^3$ in $\mathbb{F}_{2^{11}}$

$(x_0x_{10} + x_0 + x_1x_5 + x_1x_9 + x_2x_7 + x_2x_9 + x_2x_{10} + x_3x_4 + x_3x_5 + x_4x_8 + x_4x_9 + x_5x_{10} + x_6x_7 + x_6x_{10} + x_7x_8 + x_9x_{10},$

$x_0x_1 + x_0x_6 + x_2x_5 + x_2x_8 + x_3x_6 + x_3x_9 + x_3x_{10} + x_4 + x_5x_8 + x_5x_9 + x_6x_9 + x_7x_8 + x_7x_9 + x_7 + x_{10},$

$x_0x_1 + x_0x_2 + x_0x_{10} + x_1x_5 + x_1x_6 + x_1x_9 + x_2x_7 + x_3x_4 + x_3x_7 + x_4x_5 + x_4x_8 + x_4x_{10} + x_5x_{10} + x_6x_7 + x_6x_8 + x_6x_9 + x_7x_{10} + x_8 + x_9x_{10},$

$x_0x_3 + x_0x_6 + x_0x_7 + x_1 + x_2x_5 + x_2x_6 + x_2x_8 + x_2x_{10} + x_3x_6 + x_3x_8 + x_3x_9 + x_4x_5 + x_4x_6 + x_4 + x_5x_8 + x_5x_{10} + x_6x_9 + x_7x_9 + x_7 + x_8x_9 + x_{10},$

$x_0x_2 + x_0x_4 + x_1x_2 + x_1x_6 + x_1x_7 + x_2x_9 + x_2x_{10} + x_3x_5 + x_3x_6 + x_3x_7 + x_3x_9 + x_4x_5 + x_4x_7 + x_4x_9 + x_5 + x_6x_8 + x_7x_8 + x_8x_9 + x_8x_{10},$

$x_0x_5 + x_0x_7 + x_0x_8 + x_1x_2 + x_1x_3 + x_2x_6 + x_2x_7 + x_2x_{10} + x_3x_8 + x_4x_5 + x_4x_8 + x_5x_6 + x_5x_9 + x_7x_8 + x_7x_9 + x_7x_{10} + x_9,$

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Algebraic degree - 2nd definition

Let $F : \mathbb{F}_2^n \to \mathbb{F}_2^n$. Then using the isomorphism $\mathbb{F}_2^n \simeq \mathbb{F}_{2^n}$,
there is **a unique univariate polynomial representation** on $\mathbb{F}_{2^n}$ of degree at most $2^n - 1$:

$$F(x) = \sum_{i=0}^{2^n-1} b_i x^i ; b_i \in \mathbb{F}_{2^n}$$

Proposition

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If $F : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^n$ is a permutation, then

$$\boxed{\deg^a(F) \leq n - 1}$$

Higher-Order differential attacks

Exploiting a low algebraic degree

For any affine subspace $\mathcal{V} \subset \mathbb{F}_2^n$ with $\dim \mathcal{V} \geq \deg^a(F) + 1$, we have a 0-sum distinguisher:

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Random permutation: degree $= n - 1$

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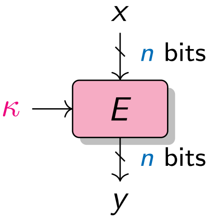
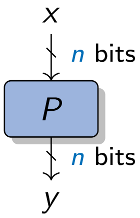
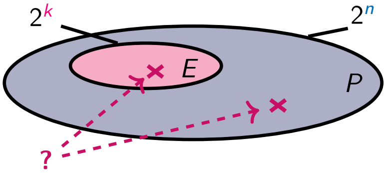


(a) *Block cipher*



(b) *Random permutation*



First Plateau

Polynomial representing r rounds of MIMC$_d$:

$$\mathcal{P}_{d,r}(x) = F_r \circ \ldots F_1(x) \ , \ \text{where } F_i = (x + c_{i-1})^d \ .$$

Upper bound [Eichlseder et al., AC20]:

$$\lceil r \log_2 d \rceil \ .$$

Aim: determine

$$B_d^r := \max_c \deg^a(\mathcal{P}_{d,r}) \ .$$

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Polynomial representing r rounds of MIMC$_d$:

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Upper bound [Eichlseder et al., AC20]:

$$\lceil r \log_2 d \rceil \ .$$

Aim: determine

$$B_d^r := \max_c \deg^a(\mathcal{P}_{d,r}) \ .$$

Example: when $d = 3$

- ⋆ Round 1: $B_3^1 = 2$

$$\mathcal{P}_{3,1}(x) = x^3$$

$$3 = [11]_2$$

First Plateau

Polynomial representing r rounds of MIMC_d:

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Example: when $d = 3$

⋆ Round 1: $B_3^1 = 2$

$$\mathcal{P}_{3,1}(x) = x^3$$

$$3 = [11]_2$$

⋆ Round 2: $B_3^2 = 2$

$$\mathcal{P}_{3,2}(x) = x^9 + c_1 x^6 + c_1^2 x^3 + c_1^3$$

$$9 = [1001]_2 \ 6 = [110]_2 \ 3 = [11]_2$$

Observed degree

Definition

There is a **plateau** between rounds r and $r+1$ whenever:

$$B_d^{r+1} = B_d^r \ .$$

Proposition

If $d = 2^j - 1$, there is always **plateau** between rounds 1 and 2:

$$B_d^2 = B_d^1 \ .$$

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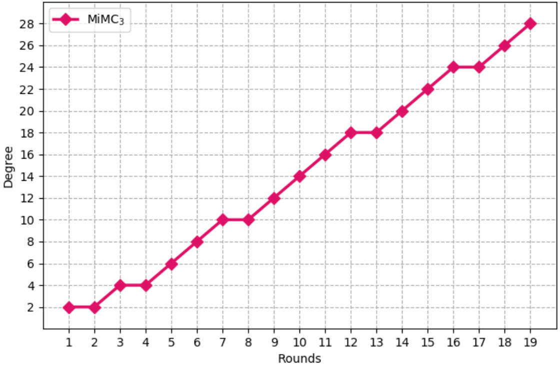

Algebraic degree observed for $n = 31$.

Missing exponents

Proposition

Set of exponents that might appear in the polynomial:

$$\mathcal{E}_{d,r} = \{d \times j \bmod (2^n - 1) \text{ where } j \text{ is covered by } i,\ i \in \mathcal{E}_{d,r-1}\}$$

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Example

$$\mathcal{P}_{3,1}(x) = x^3 \quad \text{so} \quad \mathcal{E}_{3,1} = \{3\} \ .$$

$$3 = [11]_2 \xrightarrow{\text{cover}} \begin{cases} [00]_2 = 0 & \xrightarrow{\times 3} \quad 0 \\ [01]_2 = 1 & \xrightarrow{\times 3} \quad 3 \\ [10]_2 = 2 & \xrightarrow{\times 3} \quad 6 \\ [11]_2 = 3 & \xrightarrow{\times 3} \quad 9 \end{cases}$$

$$\mathcal{E}_{3,2} = \{0, 3, 6, 9\} \ , \quad \text{indeed} \quad \mathcal{P}_{3,2}(x) = x^9 + c_1 x^6 + c_1^2 x^3 + c_1^3 \ .$$

Missing exponents

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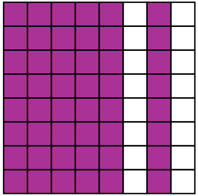
Missing exponents: no exponent $2^{2k} - 1$

Proposition

$$\forall i \in \mathcal{E}_{3,r}, i \not\equiv 5, 7 \bmod 8$$

0	1	2	3	4	5	6	7
8	9	10	11	12	13	14	15
16	17	18	19	20	21	22	23
24	25	26	27	28	29	30	31
32	33	34	35	36	37	38	39
40	41	42	43	44	45	46	47
48	49	50	51	52	53	54	55
56	57	58	59	60	61	62	63

Representation exponents.



Missing exponents mod8.

Missing exponents when $d = 2^j - 1$

- For MIMC_3

$$i \bmod 8 \notin \{5, 7\} .$$

- For MIMC_7

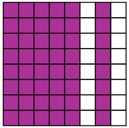
$$i \bmod 16 \notin \{9, 11, 13, 15\} .$$

- For MIMC_{15}

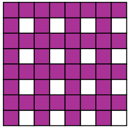
$$i \bmod 32 \notin \{17, 19, 21, 23, 25, 27, 29, 31\} .$$

- For MIMC_{31}

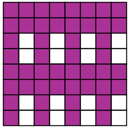
$$i \bmod 64 \notin \{33, 35, 37, 39, 41, 43, 45, 47, 49, 51, 53, 55, 57, 59, 61, 63\} .$$



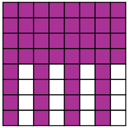
(a) *For* MIMC_3.



(b) *For* MIMC_7.



(c) *For* MIMC_{15}.



(d) *For* MIMC_{31}.

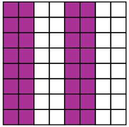
Proposition

Let $i \in \mathcal{E}_{d,r}$, where $d = 2^j - 1$. Then:

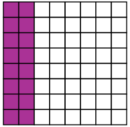
$$\forall\, i \in \mathcal{E}_{d,r}, \ i \bmod 2^{j+1} \in \{0, 1, \ldots 2^j\} \ \cup \ \{2^j + 2\gamma, \gamma = 1, 2, \ldots 2^{j-1} - 1\} .$$

Missing exponents when $d = 2^j + 1$

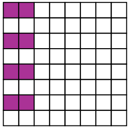
- For MIMC_5
  $$i \bmod 4 \in \{0, 1\} \ .$$
- For MIMC_9
  $$i \bmod 8 \in \{0, 1\} \ .$$
- For MIMC_{17}
  $$i \bmod 16 \in \{0, 1\} \ .$$
- For MIMC_{33}
  $$i \bmod 32 \in \{0, 1\} \ .$$



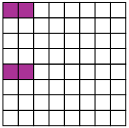
(a) *For* MIMC_5.



(b) *For* MIMC_9.



(c) *For* MIMC_{17}.



(d) *For* MIMC_{33}.

Proposition

Let $i \in \mathcal{E}_{d,r}$ where $d = 2^j + 1$ and $j > 1$. Then:

$$\forall\, i \in \mathcal{E}_{d,r}, \ \ i \bmod 2^j \in \{0, 1\} \ .$$

Bounding the degree

Theorem

After r rounds of MIMC_3, the algebraic degree is

$$B_3^r \leq 2 \times \lceil \lfloor r \log_2 3 \rfloor / 2 - 1 \rceil$$

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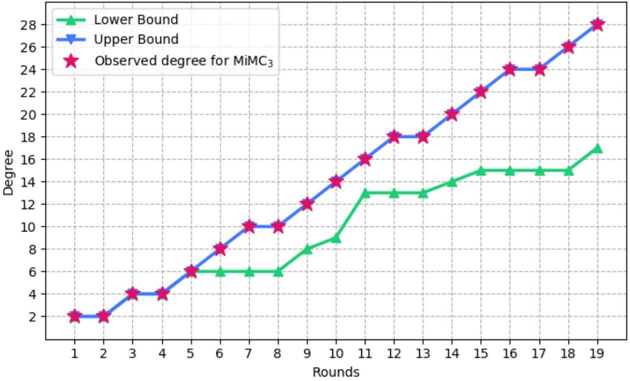
If $3^r < 2^n - 1$:

- ⋆ A lower bound

$$B_3^r \geq \max\{\mathrm{wt}(3^i), i \leq r\}$$

- ⋆ **Upper bound reached for almost 16265 rounds**

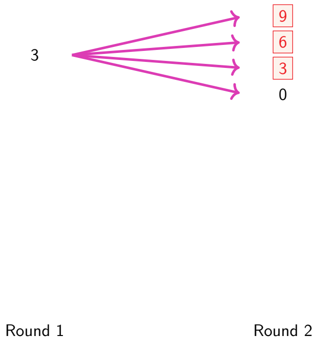


Tracing exponents

3

Round 1

Tracing exponents



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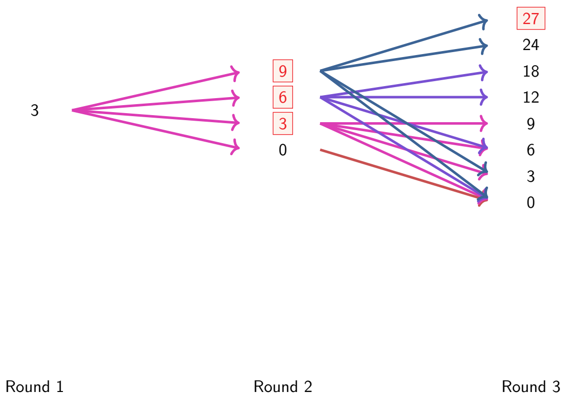

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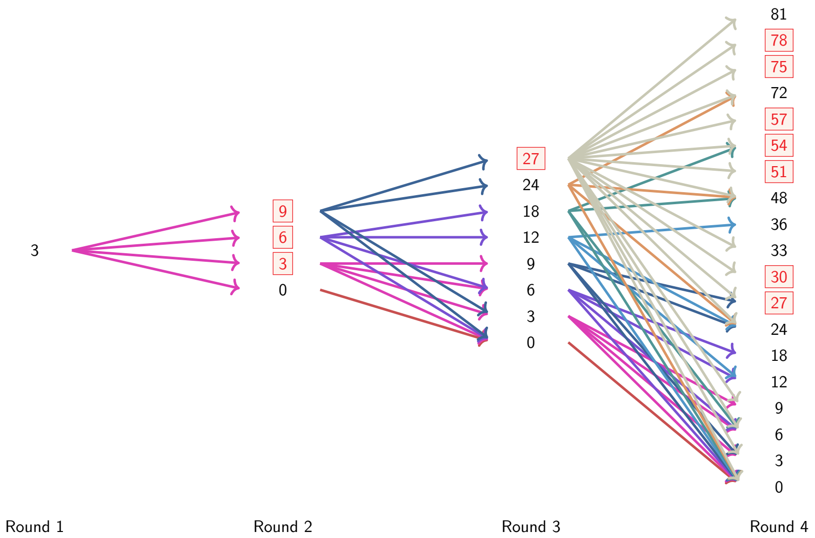

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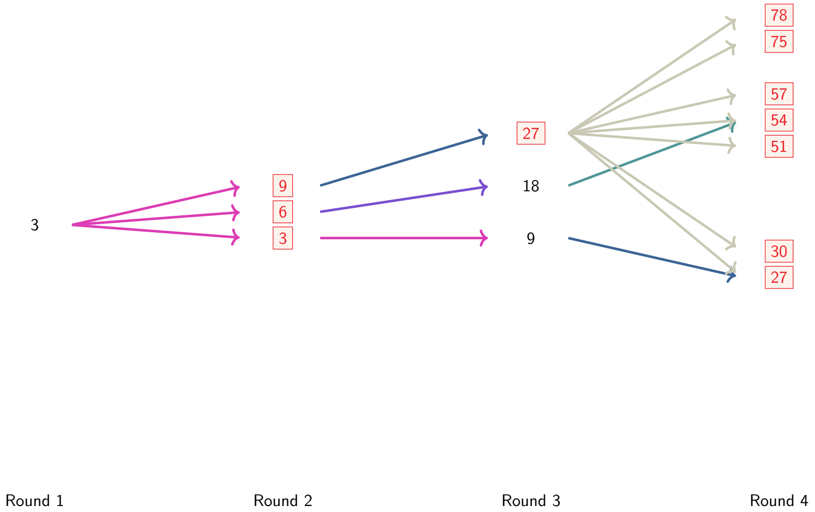

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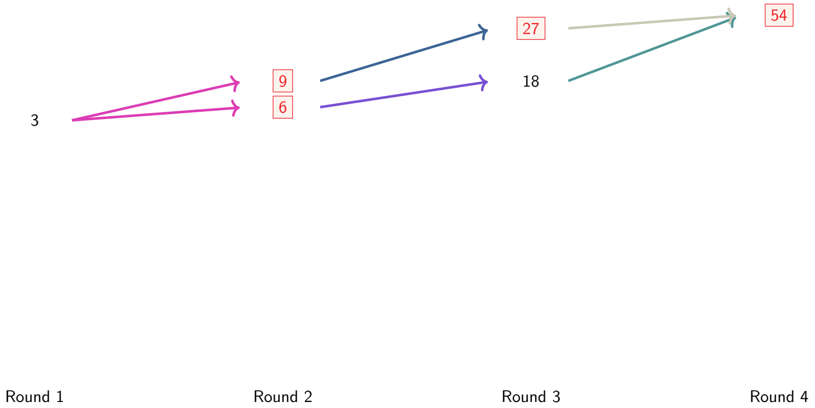

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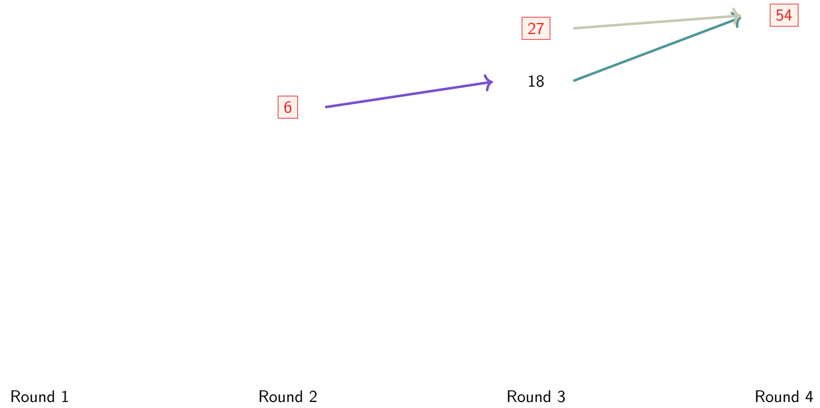

Exact degree

Maximum-weight exponents:

Let $k_r = \lfloor \log_2 3^r \rfloor$.

$\forall r \in \{4, \ldots, 16265\} \backslash \mathcal{F}$ with $\mathcal{F} = \{465, 571, \ldots\}$:

- if $k_r = 1 \bmod 2$,

$$\omega_r = 2^{k_r} - 5 \in \mathcal{E}_{3,r},$$

- if $k_r = 0 \bmod 2$,

$$\omega_r = 2^{k_r} - 7 \in \mathcal{E}_{3,r}.$$

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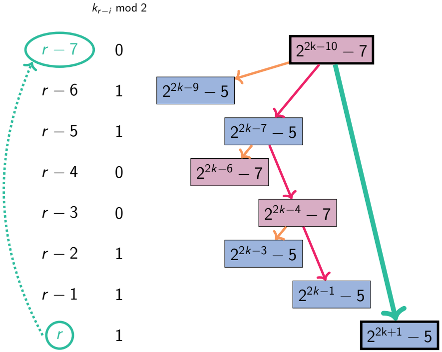

Constructing exponents.

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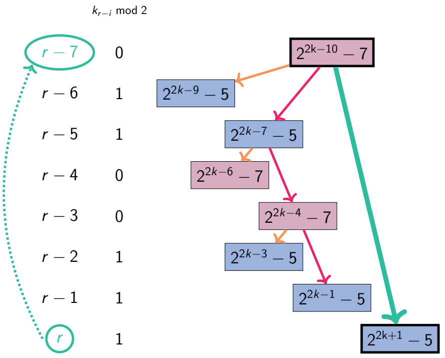


Constructing exponents.

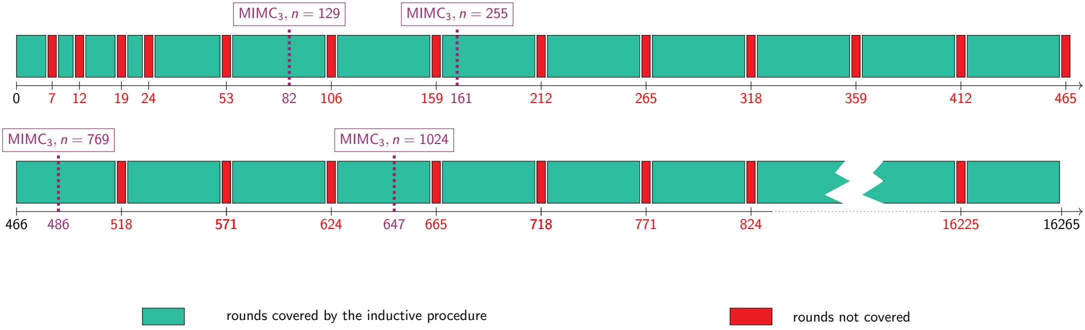
In most cases, $\exists\, \ell$ s.t. $\omega_{r-\ell} \in \mathcal{E}_{3,r-\ell} \Rightarrow \omega_r \in \mathcal{E}_{3,r}$

Covered rounds

Idea of the proof:

- ⋆ inductive proof: existence of “good” ℓ

Rounds for which we are able to exhibit a maximum-weight exponent.



Covered rounds

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- ⋆ inductive proof: existence of “good” ℓ
- ⋆ MILP solver (`PySCIPOpt`)

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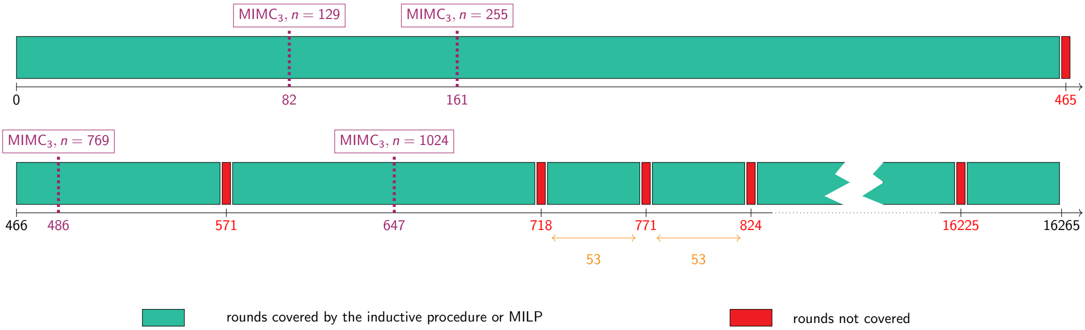
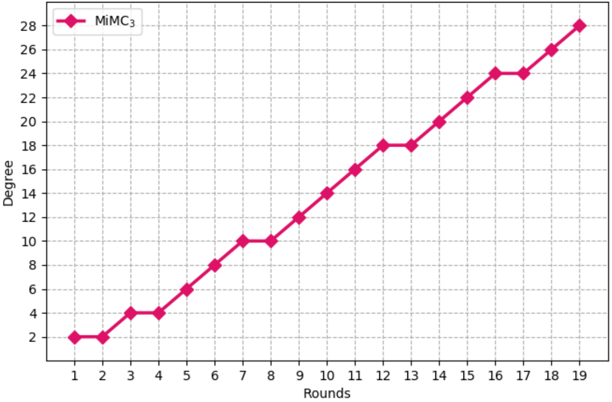


Plateau

Proposition

There is a plateau when $k_r = \lfloor r \log_2 3 \rfloor = 1 \bmod 2$ and $k_{r+1} = \lfloor (r+1) \log_2 3 \rfloor = 0 \bmod 2$



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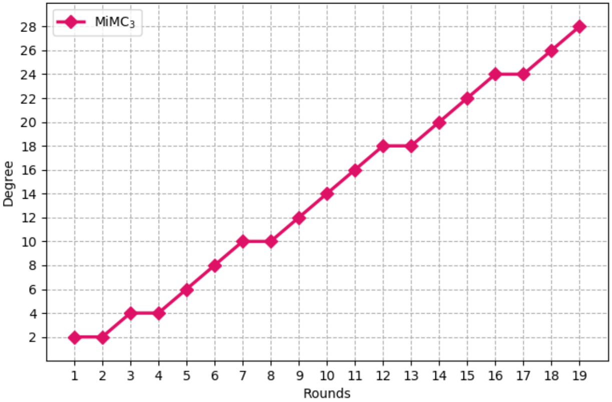


If we have a plateau

$$B_3^r = B_3^{r+1} ,$$

Then the next one is

$$B_3^{r+4} = B_3^{r+5}$$

or

$$B_3^{r+5} = B_3^{r+6} .$$

Music in MIMC_3

⋆ Patterns in sequence $(\lfloor r \log_2 3 \rfloor)_{r>0}$: denominators of semiconvergents of

$$\log_2(3) \simeq 1.5849625$$

$$\mathfrak{D} = \{\boxed{1}, \boxed{2}, 3, 5, \boxed{7}, \boxed{12}, 17, 29, 41, \boxed{53}, 94, 147, 200, 253, 306, \boxed{359}, \ldots\},$$

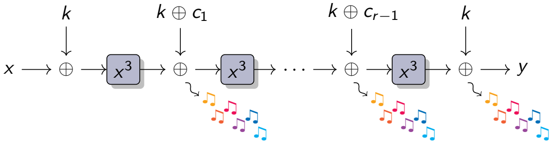
$$\log_2(3) \simeq \frac{a}{b} \quad \Leftrightarrow \quad 2^a \simeq 3^b$$

⋆ **Music theory:**

- ⋆ perfect octave 2:1
- ⋆ perfect fifth 3:2

$$2^{19} \simeq 3^{12} \quad \Leftrightarrow \quad 2^7 \simeq \left(\frac{3}{2}\right)^{12}$$

$$\Leftrightarrow \quad \text{7 octaves} \sim \text{12 fifths}$$



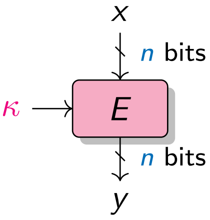
Higher-Order differential attacks

Exploiting a low algebraic degree

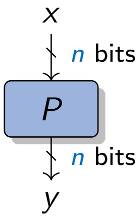
For any affine subspace $\mathcal{V} \subset \mathbb{F}_2^n$ with $\dim \mathcal{V} \geq \deg^a(F) + 1$, we have a 0-sum distinguisher:

$$\bigoplus_{x \in \mathcal{V}} F(x) = 0.$$

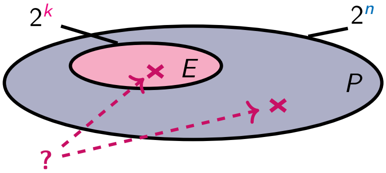
Random permutation: degree $= n - 1$



(a) *Block cipher*

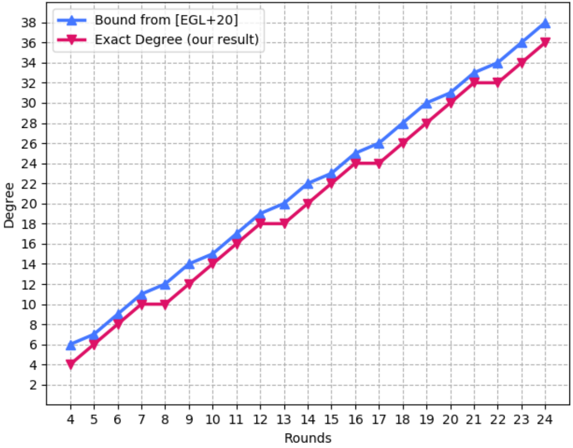


(b) *Random permutation*



Comparison to previous work

First Bound: $\lceil r \log_2 3 \rceil$ Exact degree: $2 \times \lceil \lfloor r \log_2 3 \rfloor / 2 - 1 \rceil$.



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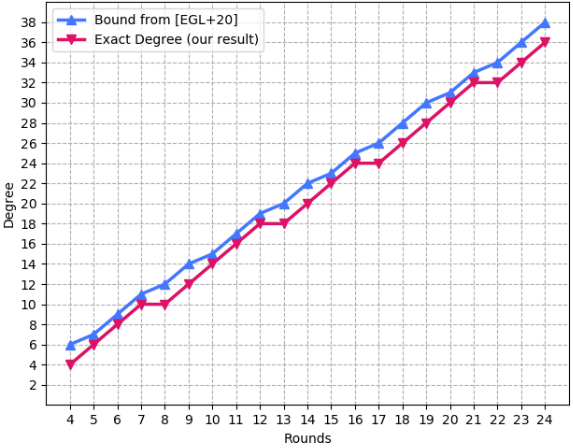

For $n = 129$, $\text{MIMC}_3 = 82$ rounds

Rounds	Time	Data	Source
80/82	2^{128}XOR	2^{128}	[EGL+20]
81/82	2^{128}XOR	2^{128}	**New**
80/82	2^{125}XOR	2^{125}	**New**

Secret-key distinguishers ($n = 129$)

Take-Away

A better understanding of the algebraic degree of MiMC

- ⋆ guarantee on the degree of MIMC_3
  - ⋆ upper bound on the algebraic degree

$$2 \times \lceil \lfloor r \log_2 3 \rfloor / 2 - 1 \rceil .$$

  - ⋆ bound tight, up to 16265 rounds
- ⋆ minimal complexity for higher-order differential attack

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Missing exponents in the univariate representation

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↓

Bounds on the algebraic degree

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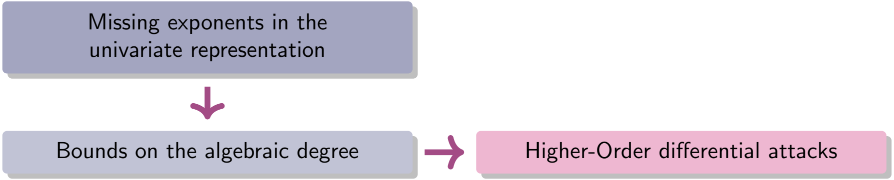

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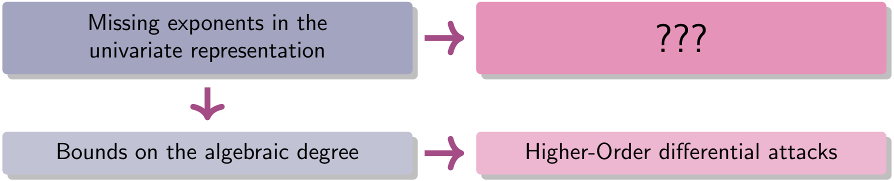

Conclusions

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- New tools for designing primitives:
  - Anemoi: a new family of ZK-friendly hash functions
  - a link between CCZ-equivalence and AO
  - more general contributions: Jive, Flystel

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- New tools for designing primitives:
  - `Anemoi`: a new family of ZK-friendly hash functions
  - a link between CCZ-equivalence and AO
  - more general contributions: `Jive`, `Flystel`
- Practical and theoretical cryptanalysis
  - a better insight into the behaviour of algebraic systems
  - a comprehensive understanding of the univariate representation of MiMC
  - guarantees on the algebraic degree of MiMC

Perspectives

- On the design
  - a Flystel with more branches
  - solve the conjecture for the linearity

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Cryptanalysis and designing of arithmetization-oriented primitives remain to be explored!

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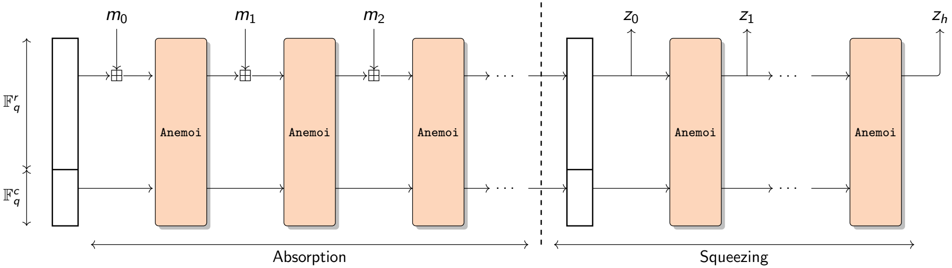
Cryptanalysis and designing of arithmetization-oriented primitives remain to be explored!

Anemoi

More benchmarks and Cryptanalysis

Sponge construction

- Hash function (random oracle):
  - input: arbitrary length
  - ouput: fixed length

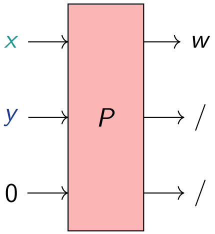


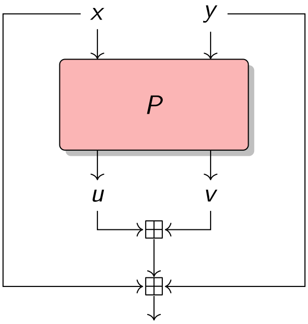
New Mode: Jive

- Compression function (Merkle-tree):
  - input: fixed length
  - output: (input length) /2

Dedicated mode: 2 words in 1

$$(x, y) \mapsto x + y + u + v\ .$$



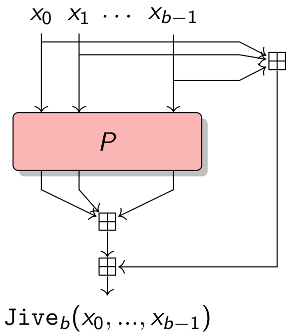


New Mode: Jive

- Compression function (Merkle-tree):
  - input: fixed length
  - output: (input length) /b

Dedicated mode: b words in 1

$$\texttt{Jive}_b(P) : \begin{cases} (\mathbb{F}_q^m)^b & \to \mathbb{F}_q^m \\ (x_0, \dots, x_{b-1}) & \mapsto \sum_{i=0}^{b-1} (x_i + P_i(x_0, \dots, x_{b-1})) \end{cases} .$$



Comparison for Plonk (with optimizations)

	m	Constraints
Poseidon	3	110
	2	88
Reinforced Concrete	3	378
	2	236
Rescue–Prime	3	252
Griffin	3	125
AnemoiJive	2	~~86~~ 56

(a) *With 3 wires.*

	m	Constraints
Poseidon	3	98
	2	82
Reinforced Concrete	3	267
	2	174
Rescue–Prime	3	168
Griffin	3	111
AnemoiJive	2	**64**

(b) *With 4 wires.*

Constraints comparison with an additional custom gate for x^α. ($s = 128$).

with an additional quadratic custom gate: 56 constraints

Native performance

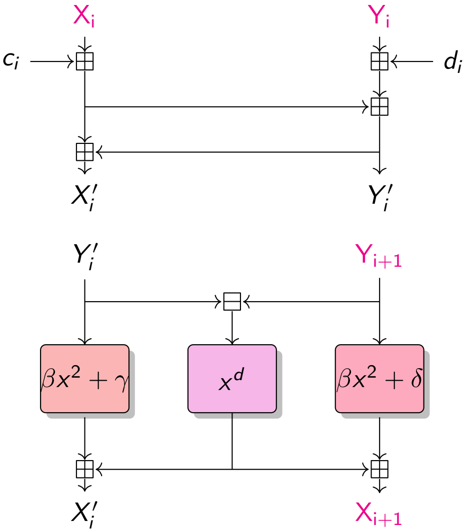
Rescue-12	*Rescue*-8	Poseidon-12	Poseidon-8	Griffin-12	Griffin-8	Anemoi-8
15.67 μs	9.13 μs	5.87 μs	2.69 μs	2.87 μs	**2.59 μs**	4.21 μs

2-to-1 compression functions for $\mathbb{F}_p$ *with* $p = 2^{64} - 2^{32} + 1$ *(*$s = 128$*).*

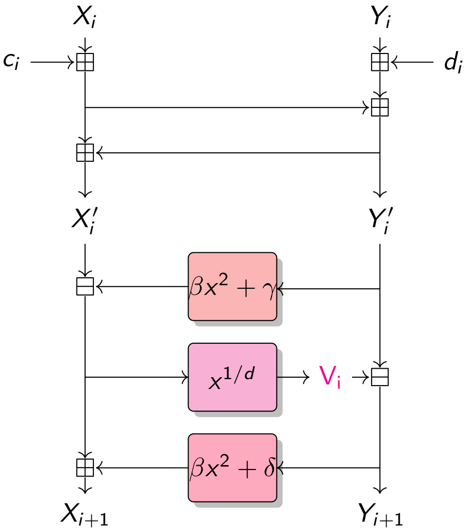
Rescue	Poseidon	Griffin	Anemoi
206 μs	**9.2 μs**	74.18 μs	128.29 μs

For $BLS12-381$, *Rescue*, Poseidon, Anemoi *with state size of* 2, Griffin *of* 3 *(*$s = 128$*).*

Algebraic attacks: 2 modelings



(a) *Model 1.*

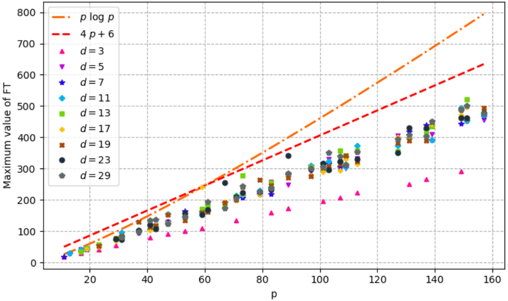


(b) *Model 2.*

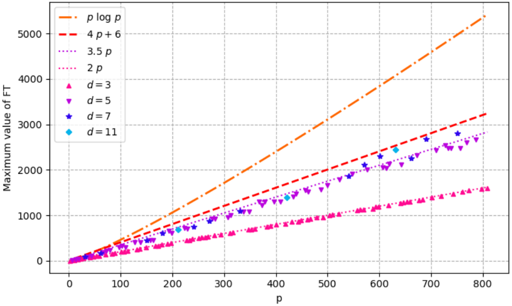
Properties of Flystel in $\mathbb{F}_p$

⋆ Linear properties

$$\mathcal{W}_{\mathcal{H}} = \max_{a,b\neq 0} \left| \sum_{x\in\mathbb{F}_p^2} exp\left(\frac{2\pi i(\langle a,x\rangle - \langle b,\mathcal{H}(x)\rangle)}{p}\right) \right| \leq p \log p \ ?$$



(a) *For different d.*



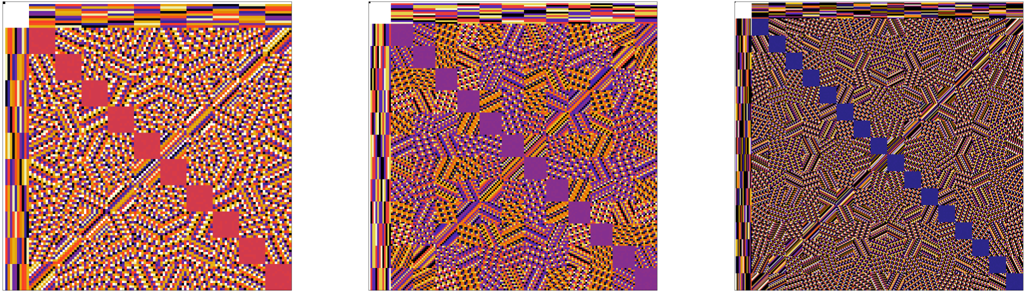
(b) *For the smallest d.*

Conjecture for the linearity.

Properties of Flystel in $\mathbb{F}_p$

- Linear properties

$$\mathcal{W}_{\mathcal{H}} = \max_{a,b \neq 0} \left| \sum_{x \in \mathbb{F}_p^2} exp \left(\frac{2\pi i(\langle a, x \rangle - \langle b, \mathcal{H}(x) \rangle)}{p} \right) \right| \leq p \log p \ ?$$



(a) *when* $p = 11$ *and* $d = 3$.

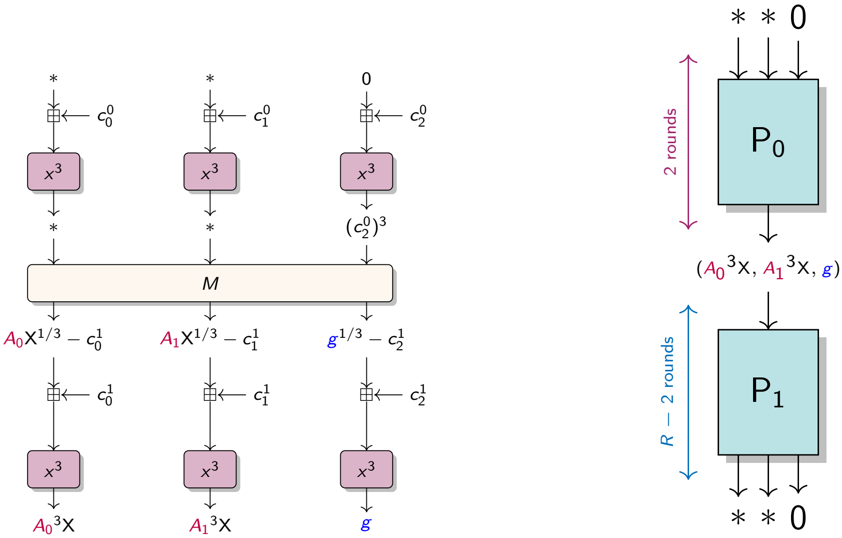
(b) *when* $p = 13$ *and* $d = 5$.

(c) *when* $p = 17$ *and* $d = 3$.

LAT of $\texttt{Flystel}_p$.

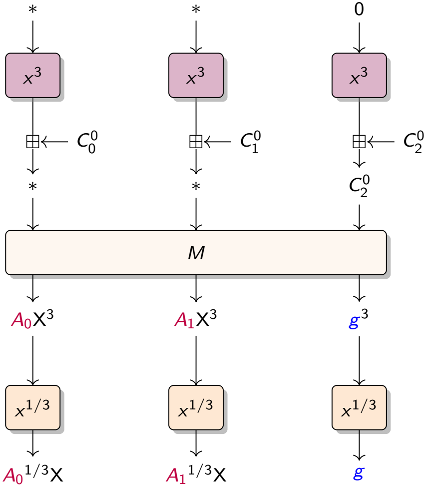
Algebraic attacks

Trick for Poseidon

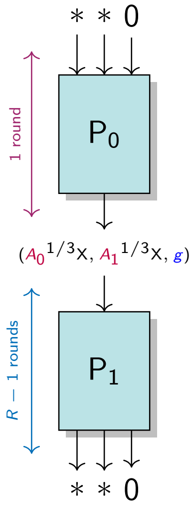


(a) *First two rounds.* **(b)** *Overview.*

Trick for Rescue–Prime



(a) *First round.*



(b) *Overview.*

Attack complexity

RP	Authors claims	Ethereum claims	$\deg^u$	Our complexity
3	2^{17}	2^{45}	$3^9 \approx 2^{14.3}$	2^{26}
8	2^{25}	2^{53}	$3^{14} \approx 2^{22.2}$	2^{35}
13	2^{33}	2^{61}	$3^{19} \approx 2^{30.1}$	2^{44}
19	2^{42}	2^{69}	$3^{25} \approx 2^{39.6}$	2^{54}
24	2^{50}	2^{77}	$3^{30} \approx 2^{47.5}$	2^{62}

(a) *For* Poseidon.

R	m	Authors claims	Ethereum claims	$\deg^u$	Our complexity
4	3	2^{36}	$2^{37.5}$	$3^9 \approx 2^{14.3}$	2^{43}
6	2	2^{40}	$2^{37.5}$	$3^{11} \approx 2^{17.4}$	2^{53}
7	2	2^{48}	$2^{43.5}$	$3^{13} \approx 2^{20.6}$	2^{62}
5	3	2^{48}	2^{45}	$3^{12} \approx 2^{19.0}$	2^{57}
8	2	2^{56}	$2^{49.5}$	$3^{15} \approx 2^{23.8}$	2^{72}

(b) *For Rescue–Prime.*

Cryptanalysis Challenge

Category	Parameters	Security level	Bounty
~~Easy~~	~~$N = 4, m = 3$~~	~~25~~	~~$2, 000~~
Easy	$N = 6, m = 2$	25	$4, 000
Medium	$N = 7, m = 2$	29	$6, 000
Hard	$N = 5, m = 3$	30	$12, 000
Hard	$N = 8, m = 2$	33	$26, 000

(a) *Rescue–Prime*

Category	Parameters	Security level	Bounty
~~Easy~~	~~$r = 6$~~	~~9~~	~~$2, 000~~
~~Easy~~	~~$r = 10$~~	~~15~~	~~$4, 000~~
~~Medium~~	~~$r = 14$~~	~~22~~	~~$6, 000~~
~~Hard~~	~~$r = 18$~~	~~28~~	$12, 000
~~Hard~~	~~$r = 22$~~	~~34~~	$26, 000

(b) *Feistel–MiMC*

Category	Parameters	Security level	Bounty
~~Easy~~	~~RP $= 3$~~	~~8~~	~~$2, 000~~
~~Easy~~	~~RP $= 8$~~	~~16~~	~~$4, 000~~
~~Medium~~	~~RP $= 13$~~	~~24~~	~~$6, 000~~
Hard	RP $= 19$	32	$12, 000
Hard	RP $= 24$	40	$26, 000

(c) Poseidon

Category	Parameters	Security level	Bounty
Easy	$p = 281474976710597$	24	$4, 000
Medium	$p = 72057594037926839$	28	$6, 000
Hard	$p = 18446744073709551557$	32	$12, 000

(d) `Reinforced Concrete`

Open problems

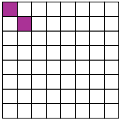
on the Algebraic Degree

Missing exponents when $d = 2^j + 1$ (first rounds)

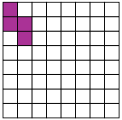
Corollary

Let $i \in \mathcal{E}_{d,r}$ where $d = 2^j + 1$ and $j > 1$. Then:

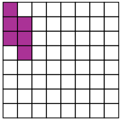
$$\begin{cases} i \bmod 2^{2j} \in \{\{\gamma 2^j, (\gamma+1)2^j + 1\},\ \gamma = 0, \ldots r-1\} & \text{if } r \leq 2^j\ , \\ i \bmod 2^j \in \{0, 1\} & \text{if } r \geq 2^j\ . \end{cases}$$



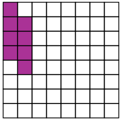
(a) *Round 1*



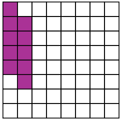
(b) *Round 2*



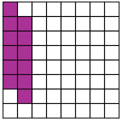
(c) *Round 3*



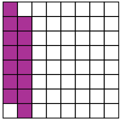
(d) *Round 4*



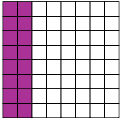
(a) *Round 5*



(b) *Round 6*



(c) *Round 7*



(d) *Round $r \geq 8$*

Bounding the degree when $d = 2^j - 1$

Note that if $d = 2^j - 1$, then

$$2^i \bmod d \equiv 2^{i \bmod j} .$$

Proposition

Let $d = 2^j - 1$, such that $j \geq 2$. Then,

$$B_d^r \leq \lfloor r \log_2 d \rfloor - (\lfloor r \log_2 d \rfloor \bmod j) .$$

Note that if $2 \leq j \leq 7$, then

$$2^{\lfloor r \log_2 d \rfloor + 1} - 2^j - 1 > d^r .$$

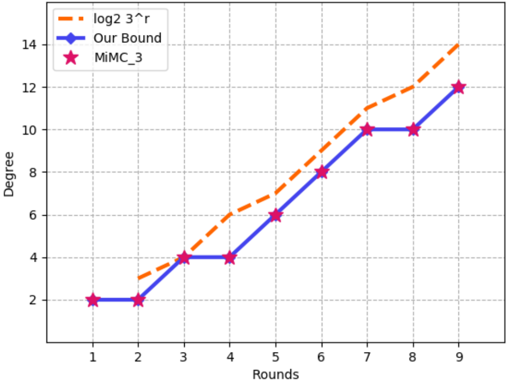
Corollary

Let $d \in \{3, 7, 15, 31, 63, 127\}$. Then,

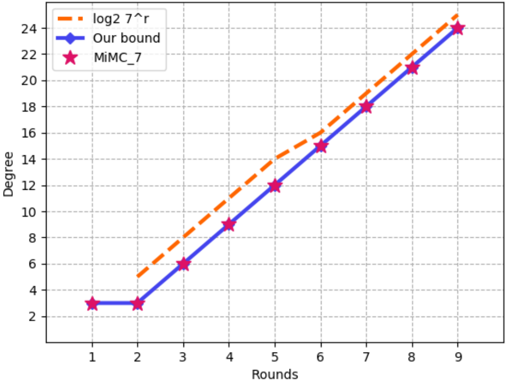
$$B_d^r \leq \begin{cases} \lfloor r \log_2 d \rfloor - j & \text{if } \lfloor r \log_2 d \rfloor \bmod j = 0 , \\ \lfloor r \log_2 d \rfloor - (\lfloor r \log_2 d \rfloor \bmod j) & \text{else} . \end{cases}$$

Bounding the degree when $d = 2^j - 1$

Particularity: Plateau when $\lfloor r \log_2 d \rfloor \bmod j = j - 1$ and $\lfloor (r+1) \log_2 d \rfloor \bmod j = 0$.



Bound for MIMC_3



Bound for MIMC_7

Bounding the degree when $d = 2^j + 1$

Note that if $d = 2^j + 1$, then

$$2^i \bmod d \equiv \begin{cases} 2^{i \bmod 2j} & \text{if } i \equiv 0, \dots, j \bmod 2j \ , \\ d - 2^{(i \bmod 2j) - j} & \text{if } i \equiv 0, \dots, j \bmod 2j \ . \end{cases}$$

Proposition

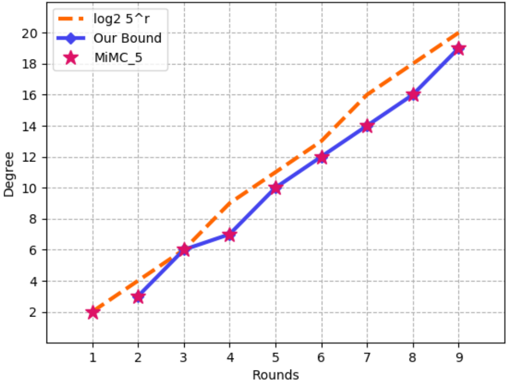
Let $d = 2^j + 1$ s.t. $j > 1$. Then if $r > 1$:

$$B_d^r \leq \begin{cases} \lfloor r \log_2 d \rfloor - j + 1 & \text{if } \lfloor r \log_2 d \rfloor \bmod 2j \in \{0, j-1, j+1\} \ , \\ \lfloor r \log_2 d \rfloor - j & \text{else} \ . \end{cases}$$

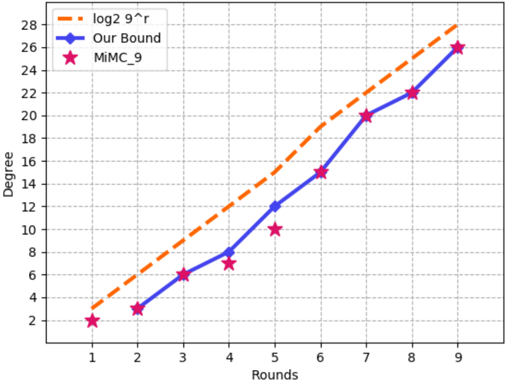
The bound can be refined on the first rounds!

Bounding the degree when $d = 2^j + 1$

Particularity: There is a gap in the first rounds.



Bound for MIMC_5



Bound for MIMC_9

Sporadic Cases

Observation

Let $k_{3,r} = \lfloor r \log_2 3 \rfloor$. If $4 \leq r \leq 16265$, then

$$3^r > 2^{k_{3,r}} + 2^r \,.$$

Observation

Let t be an integer s.t. $1 \leq t \leq 21$. Then

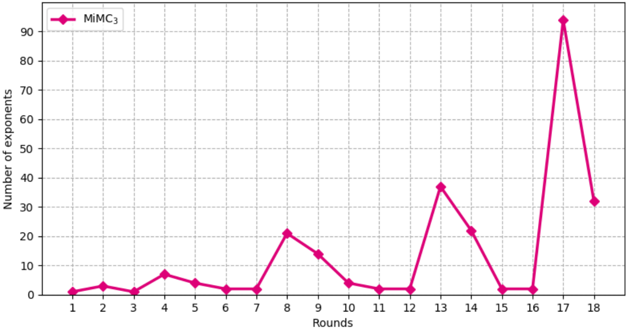
$$\forall x \in \mathbb{Z}/3^t\mathbb{Z},\ \exists \varepsilon_2, \ldots, \varepsilon_{2t+2} \in \{0,1\},\ \text{s.t. } x = \sum_{j=2}^{2t+2} \varepsilon_j 4^j \bmod 3^t \,.$$

Is it true for any t?

Should we consider more ε_j for larger t?

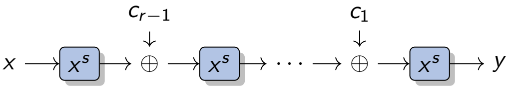
More maximum-weight exponents

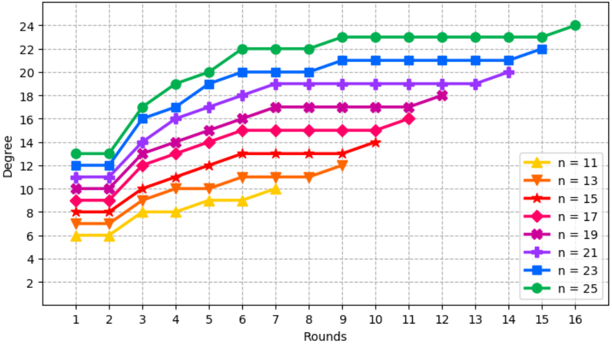
r	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$k_{3,r}$	1	3	4	6	7	9	11	12	14	15	17	19	20	22	23	25	26	28
$b_{3,r}$	1	1	0	0	1	1	1	0	0	1	1	1	0	0	1	1	0	0



Study of MiMC_3^{-1}

Inverse: $F : x \mapsto x^s, \; s = (2^{n+1} - 1)/3 = [101..01]_2$





First plateau

Plateau between rounds 1 and 2, for $s = (2^{n+1} - 1)/3 = [101..01]_2$

- ⋆ Round 1:

$$B_s^1 = \mathrm{wt}(s) = (n+1)/2$$

- ⋆ Round 2:

$$B_s^2 = \max\{\mathrm{wt}(is),\ \text{for } i \preceq s\} = (n+1)/2$$

Proposition

For $i \preceq s$ such that $\mathrm{wt}(i) \geq 2$:

$$\mathrm{wt}(is) \in \begin{cases} [\mathrm{wt}(i) - 1, (n-1)/2] & \text{if } \ wt(i) \equiv 2 \bmod 3 \\ [\mathrm{wt}(i), (n+1)/2] & \text{if } \ wt(i) \equiv 0, 1 \bmod 3 \end{cases}$$

Next Rounds

Proposition [Boura and Canteaut, IEEE13]

$\forall i \in [1, n-1]$, if the algebraic degree of encryption is $\deg^a(F) < (n-1)/i$, then the algebraic degree of decryption is $\deg^a(F^{-1}) < n - i$

$$r_{n-i} \geq \left\lceil \frac{1}{\log_2 3} \left(2 \left\lceil \frac{1}{2} \left\lceil \frac{n-1}{i} \right\rceil \right\rceil + 1 \right) \right\rceil$$

In particular:

$$r_{n-2} \geq \left\lceil \frac{1}{\log_2 3} \left(2 \left\lceil \frac{n-1}{4} \right\rceil + 1 \right) \right\rceil$$