Butterfly Constructions:

From Boolean Foundations to Remaining Challenges over Prime Fields.

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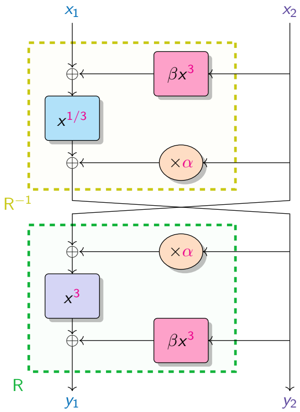
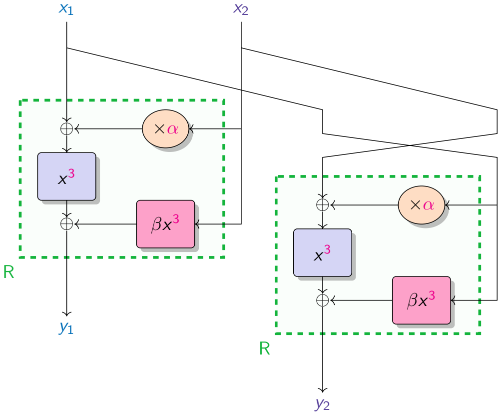
Butterfly



Open variant.



Closed variant.

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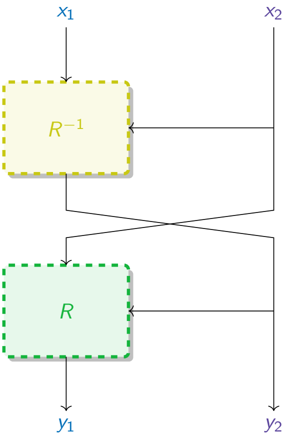

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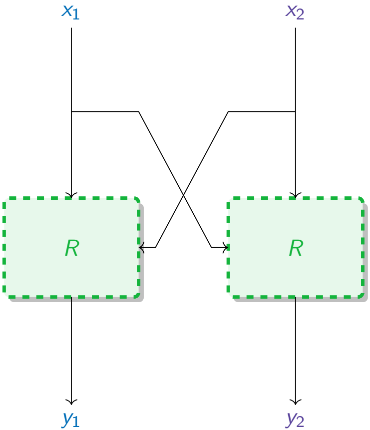

Closed variant.

Outline

From the original Butterfly construction in $\mathbb{F}_{2^n}^2$...

Preliminaries

and definitions

... to new challenges in prime fields.

Context

and recents results

Differential Uniformity

Differential uniformity

Let $F : \mathbb{F}_{2^n} \rightarrow \mathbb{F}_{2^n}$ be a function, then

$$\delta_F = \max_{a \neq 0, b} |\{x \in \mathbb{F}_{2^n}, F(x+a) + F(x) = b\}|$$

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Examples:

- If $F : x \mapsto x^{2^n - 2}$, then
$$\delta_F = \begin{cases} 4 & \text{if } n \text{ is even} \\ 2 & \text{if } n \text{ is odd} \end{cases} .$$
- If $F : x \mapsto x^{2^k + 1}$, then
$$\delta_F = 2 .$$

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⋆ If $F : x \mapsto x^{2^k+1}$, then

$$\delta_F = 2 .$$

APN (Almost Perfect Non-linear) functions

A function F is APN if for all $a \neq 0$ and b, we have $\delta_F \leq 2$.

Linearity

Linearity

Let $F : \mathbb{F}_{2^n} \rightarrow \mathbb{F}_{2^n}$ be a function, then

$$\mathcal{W}_F = \max_{u,v \neq 0} \left| \sum_{x \in \mathbb{F}_{2^n}} (-1)^{u \cdot x + v \cdot F(x)} \right|$$

Correlation

The maximum correlation for a linear approximation (u, v) is

$$C_F = 2^{-n} \cdot \mathcal{W}_F$$

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Examples:

- If $F : x \mapsto Lx + c$, then

$$\mathcal{W}_F = 2^n \quad \text{and} \quad C_F = 1 \ .$$

- If $F : x \mapsto x^{-1}$, with n even, then

$$\mathcal{W}_F = 2^{n/2+1} \quad \text{and} \quad C_F = 2^{-n/2+1} \ .$$

CCZ-equivalence

Inversion

$$\Gamma_F = \{(x, F(x)), x \in \mathbb{F}_{2^n}\} \quad \text{and} \quad \Gamma_{F^{-1}} = \{(y, F^{-1}(y)), y \in \mathbb{F}_{2^n}\}$$

Noting that

$$\Gamma_F = \{(F^{-1}(y), y), y \in \mathbb{F}_{2^n}\} ,$$

then, we have:

$$\Gamma_F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Gamma_{F^{-1}} .$$

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Definition [Carlet, Charpin and Zinoviev, 1998]

$F : \mathbb{F}_{2^n} \to \mathbb{F}_{2^n}$ and $G : \mathbb{F}_{2^n} \to \mathbb{F}_{2^n}$ are **CCZ-equivalent** if

$$\Gamma_F \ = \ \mathcal{L}(\Gamma_G) + c \ , \quad \text{where } \mathcal{L} \text{ is linear.}$$

Advantages of CCZ-equivalence

If $F : \mathbb{F}_{2^n} \to \mathbb{F}_{2^n}$ and $G : \mathbb{F}_{2^n} \to \mathbb{F}_{2^n}$ are **CCZ-equivalent**. Then

⋆ **Differential** properties are the same: $\delta_F = \delta_G$.

Differential uniformity

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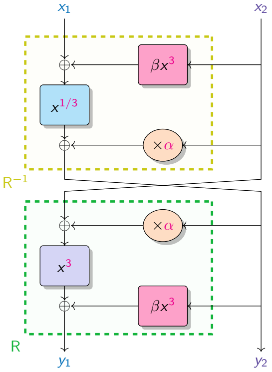
- ⋆ **Linear** properties are the same: $\mathcal{W}_F = \mathcal{W}_G$.

Linearity

$$\mathcal{W}_F = \max_{u, v \neq 0} \left| \sum_{x \in \mathbb{F}_{2^n}} (-1)^{u \cdot x + v \cdot F(x)} \right|$$

Butterfly - Definition

Introduced by [Perrin, Udovenko and Biryukov, 2016] over binary fields, $\mathbb{F}_{2^n}^2$, n odd.



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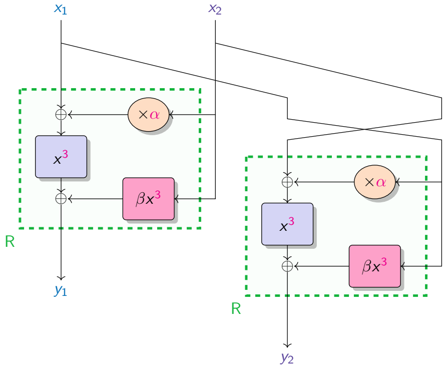

Closed variant.

$$\begin{cases} y_1 &= (x_2 + \alpha y_2)^3 + (\beta y_2)^3 \\ y_2 &= (x_1 - (\beta x_2)^3)^{1/3} - \alpha x_2 \,. \end{cases}$$

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Take-away

- ⋆ Butterfly introduced over binary fields
- ⋆ Structure of APN permutations on an even number of bits
- ⋆ 2 variants of the construction: Open and Closed
- ⋆ An example of CCZ-equivalent functions
- ⋆ Same differential and linear properties for the 2 variants

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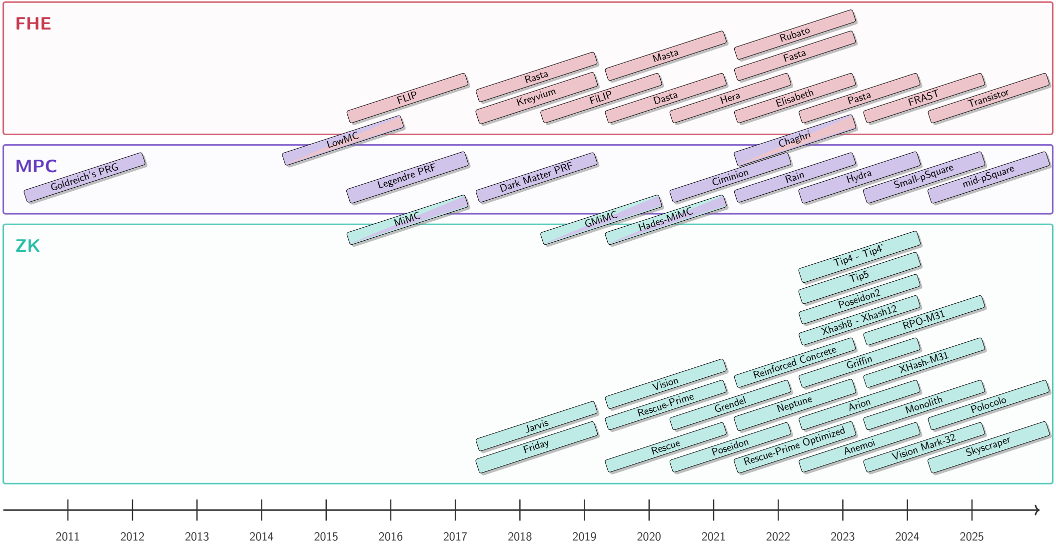
New symmetric primitives



A new context

Traditional case

Alphabet

Operations based on logical gates or CPU instructions.

$$\mathbb{F}_2^n, \text{ with } n \simeq 4, 8$$

Arithmetization-Oriented

Alphabet

Operations based on large finite-field arithmetic.

$$\mathbb{F}_q, \text{ with } q \in \{2^n, p\}, p \simeq 2^n, n \geq 32$$

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$$\mathbb{F}_2^n, \text{ with } n \simeq 4, 8$$

Cryptanalysis

Decades of cryptanalysis

- ⋆ algebraic attacks ✓
- ⋆ differential attacks ✓
- ⋆ linear attacks ✓
- ⋆ ...

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$\leq$ 8 years of cryptanalysis

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- ⋆ ...

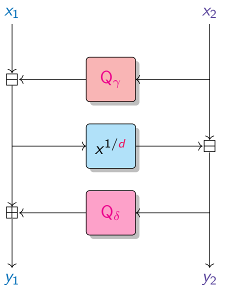
The Flystel in Anemoi

Introduced by [Bouvier, Briaud, Chaidos, Perrin, Salen, Velichkov and Willems, 2023]

Butterfly + Feistel ⇒ Flystel

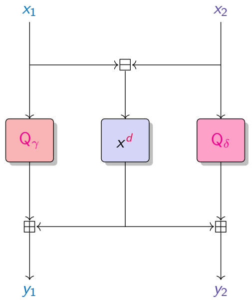
$Q_\gamma : \mathbb{F}_q \to \mathbb{F}_q$ and $Q_\delta : \mathbb{F}_q \to \mathbb{F}_q$ two quadratic functions, and $E : \mathbb{F}_q \to \mathbb{F}_q, x \mapsto x^d$ a permutation

High-Degree permutation



Open Flystel $\mathcal{H}$.

Low-Degree function



Closed Flystel $\mathcal{V}$.

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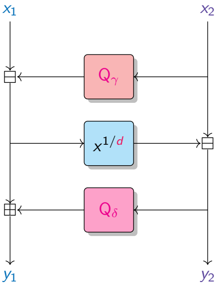

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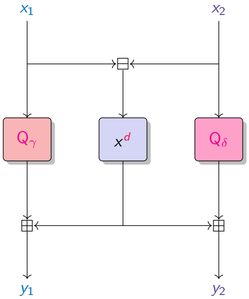

Closed Flystel $\mathcal{V}$.

$$\Gamma_{\mathcal{H}} = \mathcal{L}(\Gamma_{\mathcal{V}}) \quad \text{s.t.} \quad ((x_1, x_2), (y_1, y_2)) = \mathcal{L}\left(((y_2, x_2), (x_1, y_1)) \right)$$

How to adapt definitions

Differential uniformity

In binary fields

$$\delta_F = \max_{a \neq 0, b} \left| \{ x \in \mathbb{F}_{2^n}, F(x+a) + F(x) = b \} \right|$$

In prime fields

$$\delta_F = \max_{a \neq 0, b} \left| \{ x \in \mathbb{F}_p, F(x+a) - F(x) = b \} \right|$$

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$$\mathcal{W}_F = \max_{u, v \neq 0} \left| \sum_{x \in \mathbb{F}_{2^n}} (-1)^{u \cdot x + v \cdot F(x)} \right|$$

Characters

Definition

A **character** of a finite abelian group G is a homomorphism

$$\chi : G \rightarrow \mathbb{C}^{\times} ,$$

where $\mathbb{C}^{\times}$ is the multiplicative group of nonzero complex numbers.

In particular, we have

$$\chi(1) = 1 ,$$

and for $a_1, a_2 \in G$

$$\chi(a_1 a_2) = \chi(a_1)\chi(a_2) .$$

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Definition

A **linear approximation** of $F : \mathbb{F}_q^n \rightarrow \mathbb{F}_q^m$ is a pair of characters (χ, ψ).

Correlation of linear approximations

Definition

The **correlation of the linear approximation** (χ, ψ) of $F : \mathbb{F}_q^n \to \mathbb{F}_q^m$ is

$$C^F{}_{\chi,\psi} = \frac{1}{q^n} \sum_{x \in \mathbb{F}_q^n} \chi\big(F(x)\big)\, \psi(-x)\ .$$

Let ω be a primitive element, $\mathbb{F}_q \to \mathbb{C}^\times$ s.t. $\chi(F(x)) = \omega^{\langle v, F(x) \rangle}$ and $\psi(x) = \omega^{\langle u, x \rangle}$. Then

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Examples:

- ⋆ If $F : \mathbb{F}_2^n \to \mathbb{F}_2^m$, then

$$C^F{}_{u,v} = \frac{1}{2^n} \sum_{x \in \mathbb{F}_2^n} (-1)^{(\langle v, F(x) \rangle + \langle u, x \rangle)} \; .$$

- ⋆ If $F : \mathbb{F}_p^n \to \mathbb{F}_p^m$, then

$$C^F{}_{u,v} = \frac{1}{p^n} \sum_{x \in \mathbb{F}_p^n} e^{\left(\frac{2i\pi}{p}\right)(\langle v, F(x) \rangle - \langle u, x \rangle)} \; .$$

Walsh transform

Definition

The **Walsh transform** for the character ω of the linear approximation (u, v) of $F : \mathbb{F}_q^n \to \mathbb{F}_q^m$ is given by

$$\mathcal{W}^F{}_{u,v} = \sum_{x \in \mathbb{F}_q^n} \omega^{(\langle v, F(x) \rangle - \langle u, x \rangle)} \ .$$

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Definition

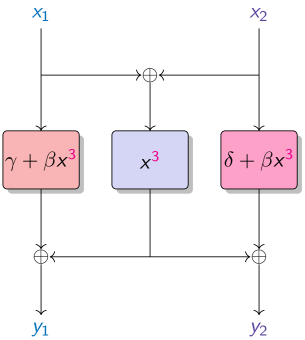
The **Linearity** $\mathcal{L}_F$ of $F : \mathbb{F}_q^n \to \mathbb{F}_q^m$ is the highest Walsh coefficient.

$$\mathcal{L}_F = \max_{u,v \in \mathbb{F}_q, v \neq 0} \left|\mathcal{W}^F{}_{u,v}\right| \ .$$

Closed Flystel in $\mathbb{F}_{2^n}$

Introduced by [Bouvier, Briaud, Chaidos, Perrin, Salen, Velichkov and Willems, 2023].

Degenerate case of Butterfly



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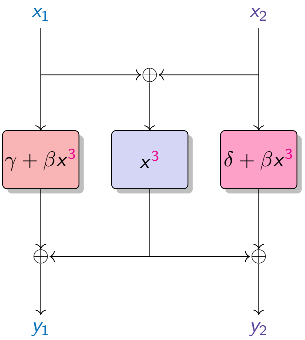


If $\beta \neq 0$, then [Li et al., 2018] stated that

Differential uniformity

$$\delta_F = \max_{a \neq 0, b} \left| \{x \in \mathbb{F}_{2^n}^2, F(x+a) + F(x) = b\} \right|$$

Bound:

$$\delta_F \leq 4$$

Linearity

$$\mathcal{L}_F = \max_{u, v \neq 0} \left| \sum_{x \in \mathbb{F}_{2^n}^2} (-1)^{(\langle v, F(x)\rangle + \langle u, x\rangle)} \right|$$

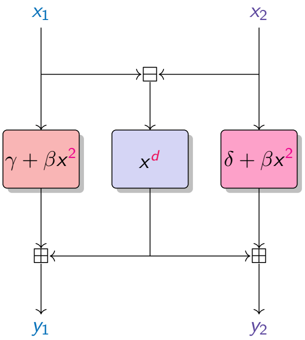
Bound:

$$\mathcal{L}_F \leq 2^{n+1}$$

Closed Flystel in $\mathbb{F}_p$

Introduced by [Bouvier, Briaud, Chaidos, Perrin, Salen, Velichkov and Willems, 2023].

$x \mapsto x^d$ a perm. (usually $d = 3, 5$)



Differential uniformity

$$\delta_F = \max_{a \neq 0, b} |\{x \in \mathbb{F}_p^2, F(x + a) - F(x) = b\}|$$

Bound:

$$\delta_F \leq d - 1$$

Solving an open problem

Finding APN permutations over $\mathbb{F}_p^2$.

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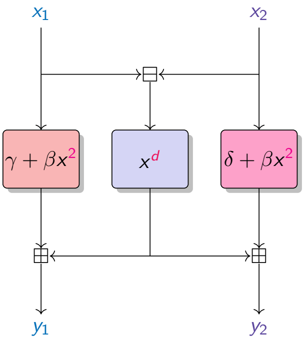

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Solving an open problem

Finding APN permutations over $\mathbb{F}_p^2$.

$$\mathcal{L}_F = \max_{u, v \neq 0} \left| \sum_{x \in \mathbb{F}_p^2} e^{\left(\frac{2i\pi}{p}\right)(\langle v, F(x)\rangle - \langle u, x\rangle)} \right|$$

How to determine an accurate bound for the linearity of the Closed Flystel in $\mathbb{F}_p$?

Weil bound

Proposition [Weil, 1948]

Let $f \in \mathbb{F}_p[x]$ be a univariate polynomial with $\deg(f) = d$. Then

$$\mathcal{L}_f \leq (d-1)\sqrt{p}$$

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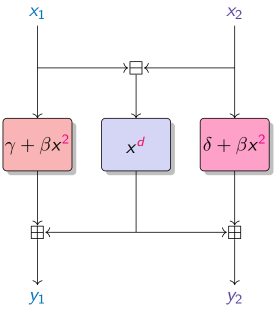


Closed Flystel.

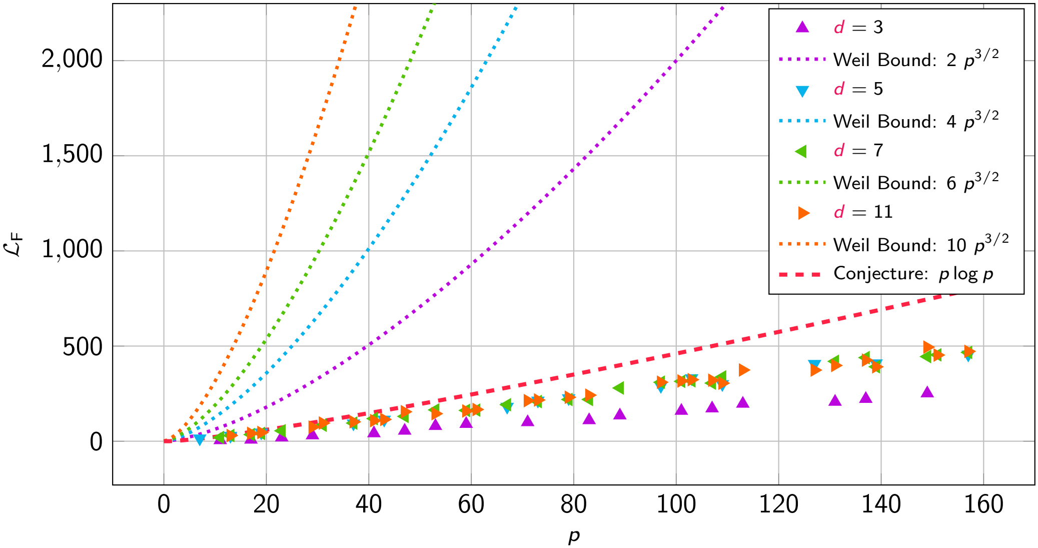
$$\mathcal{L}_F \leq (d-1)p\sqrt{p} \ ?$$

$$\begin{cases} \mathcal{L}_{\gamma+\beta x^2} & \leq \sqrt{p}\ , \\ \mathcal{L}_{x^d} & \leq (d-1)\sqrt{p}\ , \\ \mathcal{L}_{\delta+\beta x^2} & \leq \sqrt{p}\ . \end{cases}$$

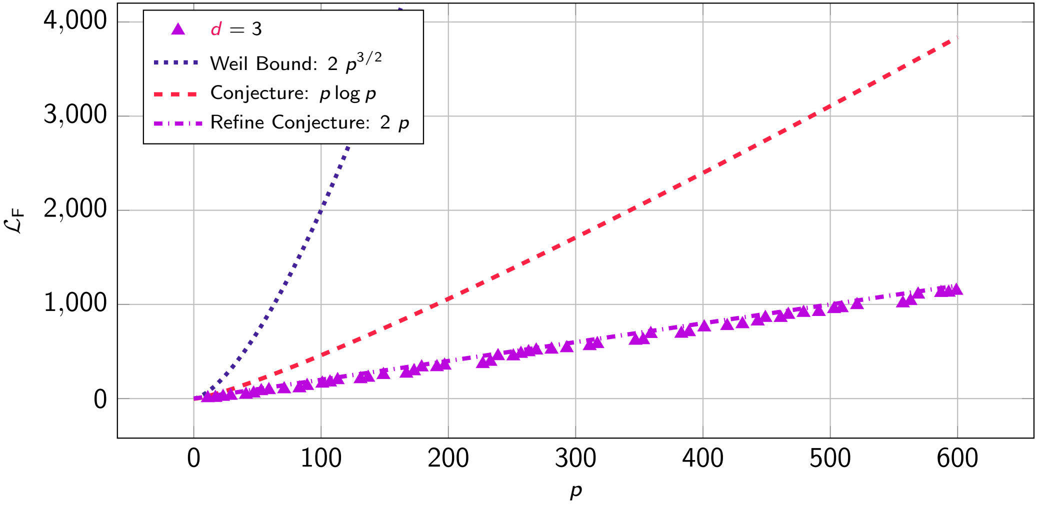
Conjecture

$$\mathcal{L}_F = \sum_{x \in \mathbb{F}_p^2} e^{\left(\frac{2i\pi}{p}\right)(\langle v, F(x)\rangle - \langle u, x\rangle)} \leq p \log p$$

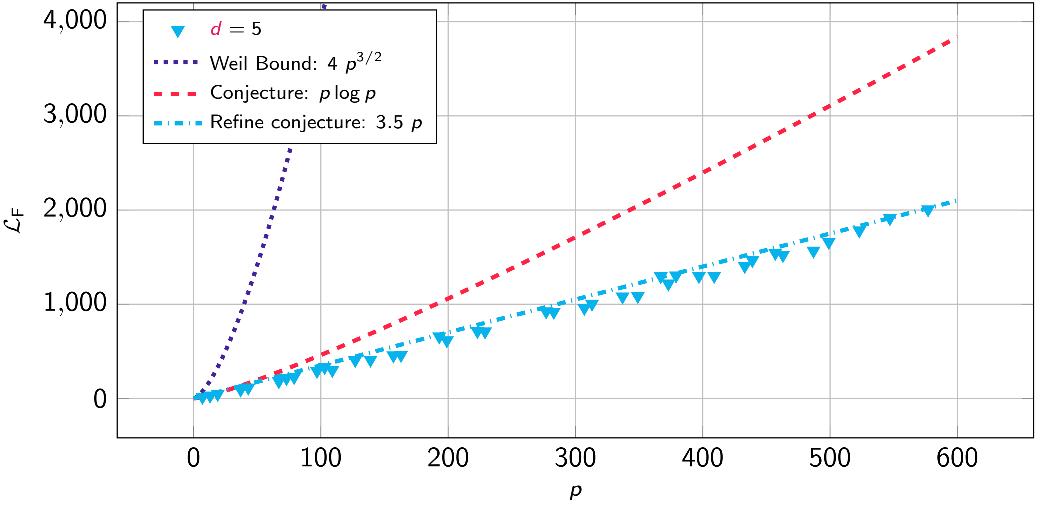
Experimental results



Experimental results ($d = 3$)



Experimental results ($d = 5$)



Take-away

AO primitives: new symmetric primitives defined over prime fields.

Need for new linear cryptanalysis tools

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This Talk:

- ⋆ Applications of results for exponential sums (generalization of Weil bound)

$$\mathcal{W}^{\mathsf{F}}_{u,v} = \sum_{x \in \mathbb{F}_q^n} \omega^{(\langle v, \mathsf{F}(x)\rangle - \langle u, x\rangle)} \quad \rightarrow \quad S(f) = \sum_{x \in \mathbb{F}_q^n} \omega^{f(x)} .$$

- ⋆ $\mathbb{F}_q$ is a finite field s.t. q is a power of a prime p.
- ⋆ Functions with 2 variables $\mathsf{F} \in \mathbb{F}_q[x_1, x_2]$.

Generalizations of Weil bound

[Beyne and Bouvier, 2024]

- ⋆ Deligne bound
  - ⋆ Application to the Generalized Butterfly construction
- ⋆ Denef and Loeser bound
  - ⋆ Application to 3-round Feistel construction
- ⋆ Rojas-León bound
  - ⋆ Application to the Generalized Flystel construction

Smoothness

Definition

Let $f \in \mathbb{F}_q[x_1, \ldots, x_n]$. A hypersurface defined by $f = 0$ is **smooth**, if the system

$$f = \partial f / \partial x_1 = \cdots = \partial f / \partial x_n = 0$$

has no non zero solutions.

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has no non zero solutions.

Examples:

- $f(x_1, x_2) = 2{x_1}^3 + {x_2}^2 = 0$ is smooth, since

$$\partial f / \partial x_1 = 6{x_1}^2 \quad \text{and} \quad \partial f / \partial x_2 = 2x_2 ,$$

so that

$$f = \partial f / \partial x_1 = \partial f / \partial x_2 = 0 \quad \Leftrightarrow \quad (x_1, x_2) = (0, 0) .$$

Smoothness

Definition

Let $f \in \mathbb{F}_q[x_1, \ldots, x_n]$. A hypersurface defined by $f = 0$ is **smooth**, if the system

$$f = \partial f / \partial x_1 = \cdots = \partial f / \partial x_n = 0$$

has no non zero solutions.

Examples:

- $f(x_1, x_2) = 2{x_1}^3 + {x_2}^2 = 0$ is smooth, since

$$\partial f / \partial x_1 = 6{x_1}^2 \qquad \text{and} \qquad \partial f / \partial x_2 = 2x_2 \,,$$

  so that

$$f = \partial f / \partial x_1 = \partial f / \partial x_2 = 0 \qquad \Leftrightarrow \qquad (x_1, x_2) = (0, 0) \,.$$

- $f(x_1, x_2) = {x_1}^2 + {x_2}^2 - 2x_2 + 1 = 0$ is not smooth, since

$$\partial f / \partial x_1 = 2x_1 \qquad \text{and} \qquad \partial f / \partial x_2 = 2x_2 - 2 \,,$$

  so that

$$f = \partial f / \partial x_1 = \partial f / \partial x_2 = 0 \qquad \Leftrightarrow \qquad (x_1, x_2) = (0, 1) \,.$$

Deligne Theorem

Theorem [Deligne, 1974]

Let q be a power of a prime p.
Let $f \in \mathbb{F}_q[x_1, \ldots, x_n]$ be a polynomial of degree d, with $\gcd(d, p) = 1$.
Let f_d be the **degree d homogeneous component** of f, i.e.

$$f = f_d + g, \ \deg(g) < d \ .$$

If the hypersurface defined by $f_d = 0$ is **smooth**, then, we have

$$|S(f)| = \left| \sum_{x \in \mathbb{F}_q^n} \omega^{f(x)} \right| \leq (d-1)^n \cdot q^{n/2} \ .$$

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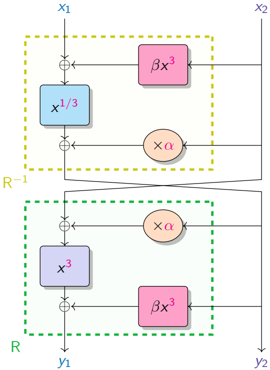
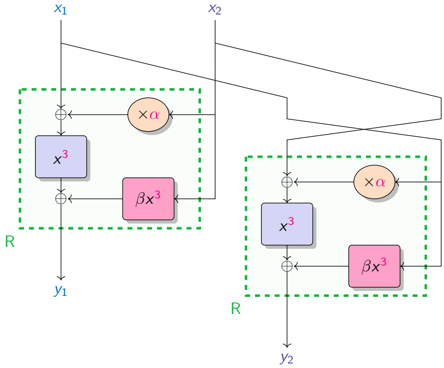
Linearity bound for $n = 2$: $\mathcal{L}_F \leq (d-1)^2 \cdot q$.

Butterfly - Definition

Introduced by [Perrin, Udovenko and Biryukov, 2016] over binary fields, $\mathbb{F}_{2^n}^2$, n odd.



Open variant.



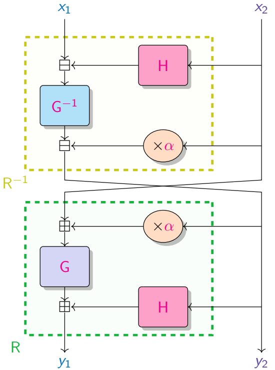
Closed variant.

$$\begin{cases} y_1 &= (x_2 + \alpha y_2)^3 + (\beta y_2)^3 \\ y_2 &= (x_1 - (\beta x_2)^3)^{1/3} - \alpha x_2 \,. \end{cases}$$

$$\begin{cases} y_1 &= (x_1 + \alpha x_2)^3 + (\beta x_2)^3 \\ y_2 &= (x_2 + \alpha x_1)^3 + (\beta x_1)^3 \,. \end{cases}$$

Generalized Butterfly - Definition

Butterfly$[G, H, \alpha]$, with $G : \mathbb{F}_q \to \mathbb{F}_q$ a permutation, $H : \mathbb{F}_q \to \mathbb{F}_q$ a function and $\alpha \in \mathbb{F}_q$.



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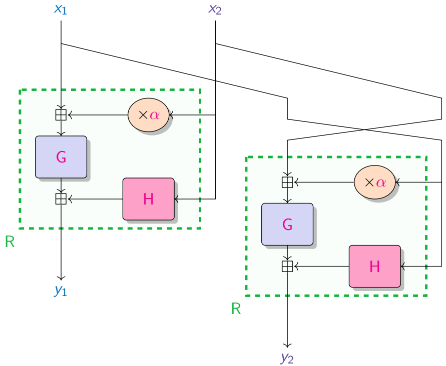

Closed variant.

$$\begin{cases} y_1 &= G(x_2 + \alpha y_2) + H(y_2) \\ y_2 &= G^{-1}(x_1 - H(x_2)) - \alpha x_2 \,. \end{cases}$$

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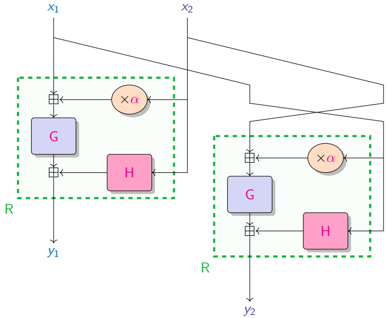
Generalized Butterfly - Bound

Let $\mathsf{F} = \text{Butterfly}[\mathsf{G}, \mathsf{H}, \alpha]$, with G a permutation, H a function and α in $\mathbb{F}_q$.

$$f(x_1, x_2) = \langle (v_1, v_2), \mathsf{F}(x_1, x_2) \rangle - \langle (u_1, u_2), (x_1, x_2) \rangle$$
$$= v_1 \mathsf{G}(x_1 + \alpha x_2) + v_2 \mathsf{G}(x_2 + \alpha x_1) + v_1 \mathsf{H}(x_2) + v_2 \mathsf{H}(x_1) - u_1 x_1 - u_2 x_2 \ .$$



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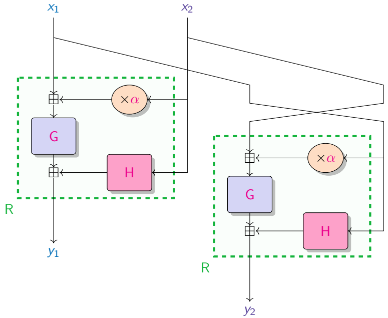

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Linearity Bound

- ⋆ If $d = \deg \mathsf{G} > \deg \mathsf{H} > 1$, then and $\alpha \neq \pm 1$,

  $f_d = (x_1 + \alpha x_2)^d + v_2 / v_1 (x_2 + \alpha x_1)^d = 0$ is smooth.

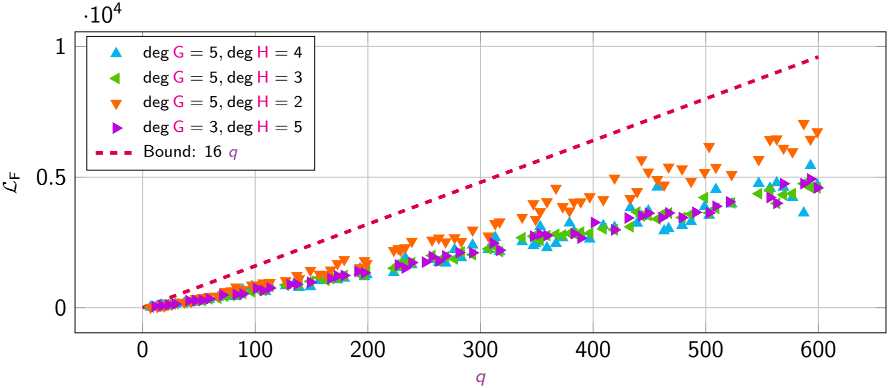
- ⋆ If $d = \deg \mathsf{H} > \deg \mathsf{G} > 1$, then

  $f_d = x_1{}^d + v_1 / v_2 x_2{}^d = 0$ is smooth.

$$\boxed{\mathcal{L}_{\mathsf{F}} \leq (\max\{\deg \mathsf{G}, \deg \mathsf{H}\} - 1)^2 \cdot q}$$

Generalized Butterfly - Results

Let $\mathsf{F} = \textsc{Butterfly}[\mathsf{G}, \mathsf{H}, \alpha]$ with G and H monomial functions.



Low-degree functions ($\max\{\deg \mathsf{G}, \deg \mathsf{H}\} = 5$ *and* $\alpha = 2$*).*

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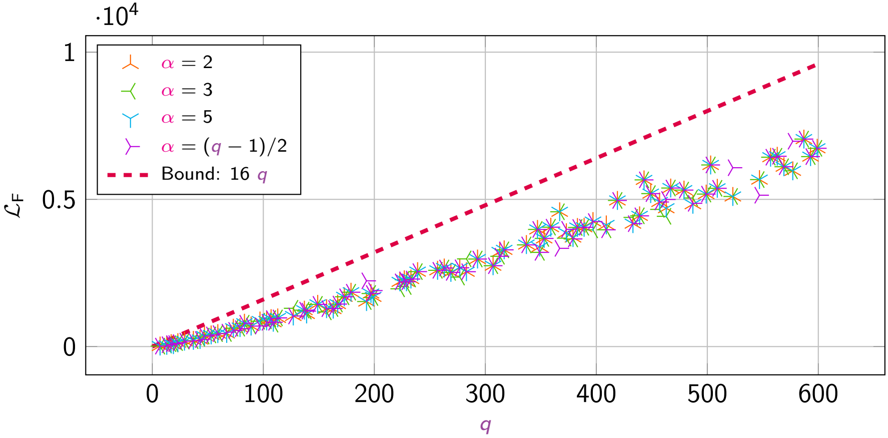
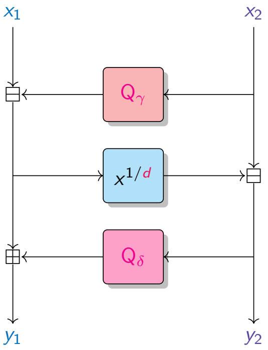


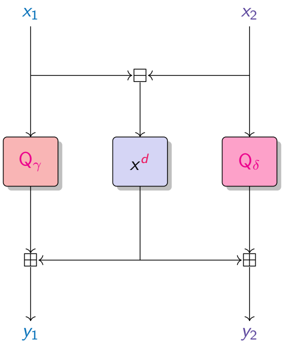
Influence of α (deg G = 5 and deg H = 2).

Flystel - Definition

Introduced by [Bouvier, Briaud, Chaidos, Perrin, Salen, Velichkov and Willems, 2023].



Open variant.



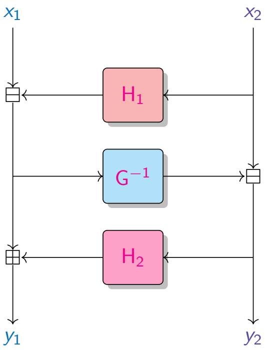
Closed variant.

$$\begin{cases} y_1 & = x_1 - Q_\gamma(x_2) + Q_\delta(x_2 - (x_1 - Q_\gamma(x_2))^{1/d}) \\ y_2 & = x_2 - (x_1 - Q_\gamma(x_2))^{1/d} . \end{cases}$$

$$\begin{cases} y_1 & = (x_1 - x_2)^d + Q_\gamma(x_1) \\ y_2 & = (x_1 - x_2)^d + Q_\delta(x_2) . \end{cases}$$

Generalized Flystel - Definition

$F = \textsc{Flystel}[H_1, G, H_2]$, with $G : \mathbb{F}_q \to \mathbb{F}_q$ a permutation, and $H_1, H_2 : \mathbb{F}_q \to \mathbb{F}_q$ functions.



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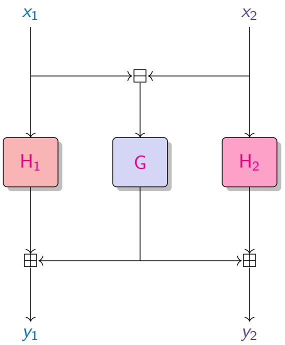

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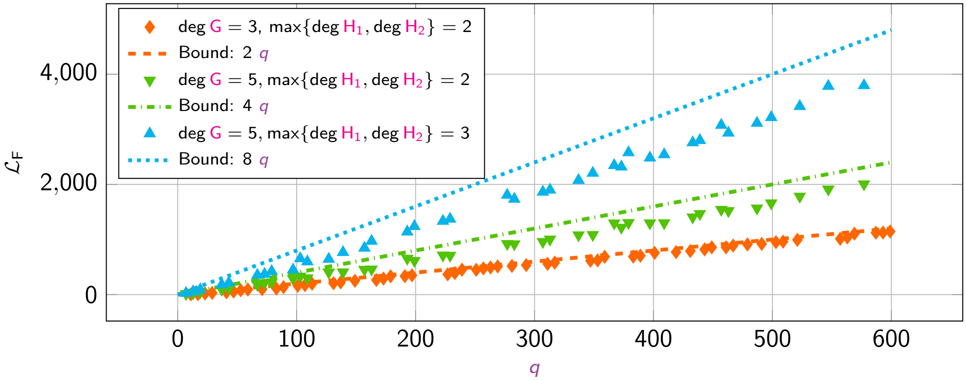
$$\begin{cases} y_1 & = x_1 - H_1(x_2) + H_2(x_2 - G^{-1}(x_1 - H_1(x_2))) \\ y_2 & = x_2 - G^{-1}(x_1 - H_1(x_2)) \,. \end{cases}$$

$$\begin{cases} y_1 & = G(x_1 - x_2) + H_1(x_1) \\ y_2 & = G(x_1 - x_2) + H_2(x_2) \,. \end{cases}$$

Generalized Flystel - Results

Let $F = \text{Flystel}[H_1, G, H_2]$ with H_1, G and H_2 monomials.

$$\mathcal{L}_F \leq (\deg G - 1)(\max\{\deg H_1, \deg H_2\} - 1) \cdot q$$

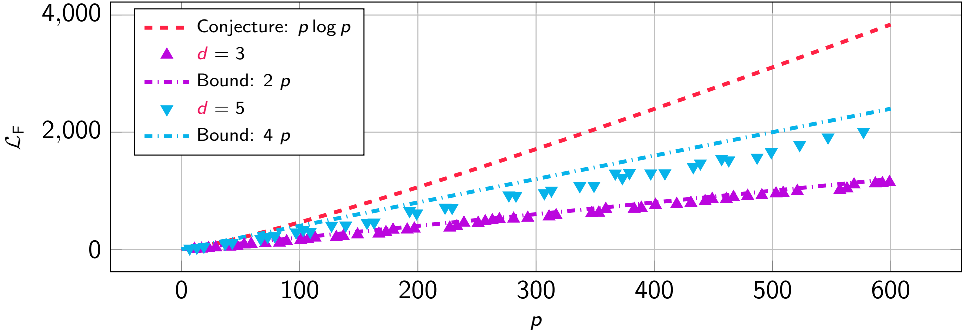


Solving conjecture

Proposition

Let $F = \text{Flystel}[H_1, G, H_2]$ be defined by $H_1(x) = \gamma + \beta x^2$, $G(x) = x^d$ and $H_2 = \delta + \beta x^2$, with $\gamma, \delta \in \mathbb{F}_p$ and $\beta \in \mathbb{F}_p^\times$. Then

$$\mathcal{L}_F \leq (d-1)p \ .$$



Conclusions

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$$\mathsf{F} \in \mathbb{F}_q[x_1, x_2],\ \exists C \in \mathbb{F}_q,\ \mathcal{L}_{\mathsf{F}} \leq C \times q$$

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Contribute to the cryptanalysis efforts for AOP.

Cohomological framework

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Sum of traces of the Frobenius automorphism on ℓ-adic cohomology groups.

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Perspectives

Can we provide detailed calculations of the cohomological spaces to refine bounds?

$$|S(f)| \leq \kappa \sum_{i=0}^{2n} \dim H_c^i(\mathbb{A}^n, \mathcal{L})$$

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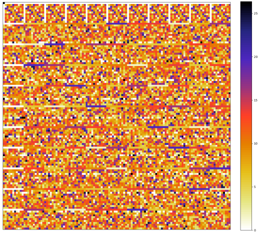
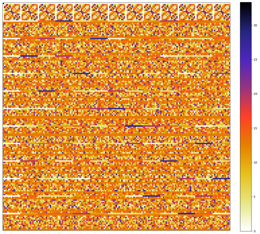


Closed Butterfly ($q = 11$)



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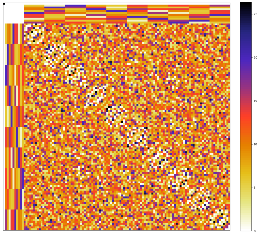

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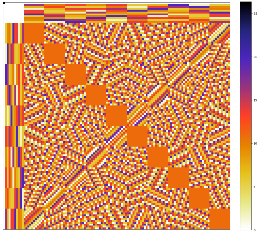

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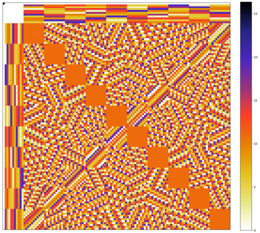

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Thank you