On the Algebraic Degree of Iterated Power Functions

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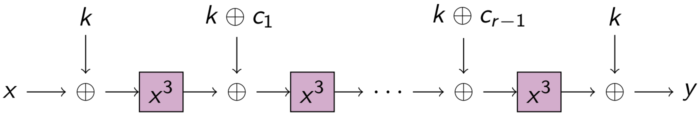
A bit of context

The block cipher MiMC

- Minimize the number of multiplications in $\mathbb{F}_{2^n}$.
- Construction of MiMC_3 [Albrecht et al., EC16]:
  - n-bit blocks (n odd ≈ 129)
  - n-bit key k
  - decryption : replacing x^3 by x^s where $s = (2^{n+1} - 1)/3$



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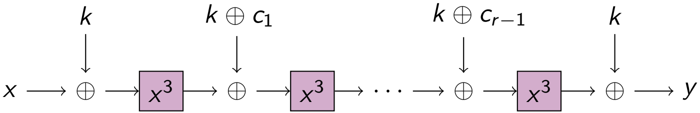
$$R := \lceil n \log_3 2 \rceil \ .$$

n	129	255	769	1025
R	82	161	486	647

Number of rounds for MiMC instances.



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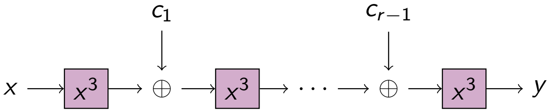

Content

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1. Background
   - Emerging uses in symmetric cryptography
   - Definition of algebraic degree

2. On the algebraic degree of MiMC_3
   - First plateau
   - Bounding the degree
   - Exact degree

3. Other permutations
   - Quadratic functions
   - Algebraic degree of MiMC_3^{-1}

4. Integral attack
   - Secret-key 0-sum distinguisher
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Emerging uses in symmetric cryptography

$\boxed{\textbf{Problem}\text{: Analyzing the security of new symmetric primitives}}$

Protocols requiring new primitives:

- multiparty computation (MPC)
- systems of zero-knowledge proofs (zk-SNARK, zk-STARK)

Primitives designed to minimize the number of multiplications in finite fields.

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"Usual" case

- operations on $\mathbb{F}_{2^n}$, where $n \simeq 4, 8$.
- based on CPU instructions and hardware components

Arithmetization-friendly

- operations on $\mathbb{F}_q$, where $q \in \{2^n, p\}, p \simeq 2^n, n \geq 64$.
- based on large finite-field arithmetic

Algebraic degree

Let $F : \mathbb{F}_2^n \to \mathbb{F}_2^n$, there is **a unique univariate polynomial representation** on $\mathbb{F}_{2^n}$ of degree at most $2^n - 1$:

$$F(x) = \sum_{i=0}^{2^n-1} b_i x^i ;\, b_i \in \mathbb{F}_{2^n}$$

Definition

Algebraic degree of $F : \mathbb{F}_2^n \to \mathbb{F}_2^n$:

$$\deg(F) = \max\{wt(i),\ 0 \le i < 2^n, \text{ and } b_i \neq 0\}$$

If $F : \mathbb{F}_2^n \to \mathbb{F}_2^n$ is a permutation, then

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Round i of MiMC_3: $x \mapsto x^3 + c_{i+1}$.

For r rounds:

- Upper bound [Eichlseder et al., AC20]: $\lceil r \log_2 3 \rceil$.
- Aim: determine $B_3^r := \max_c \deg^a \mathrm{MIMC}_{3,c}[r]$.

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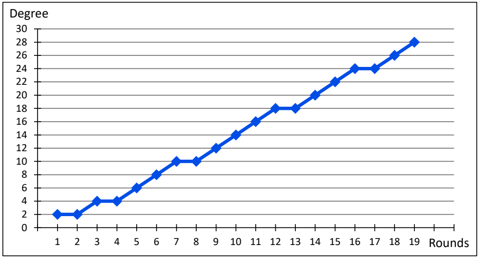

Algebraic degree observed for $n = 31$.

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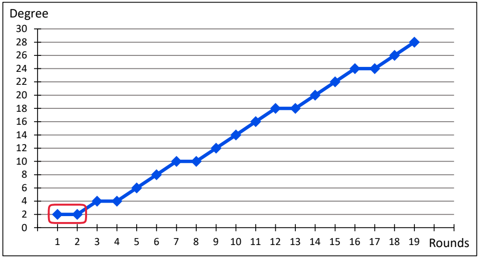

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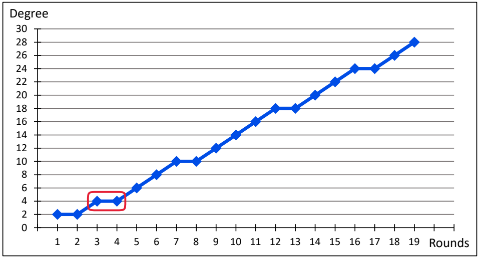

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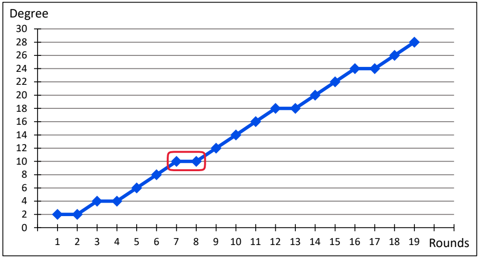

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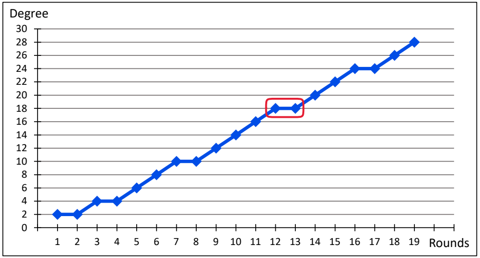

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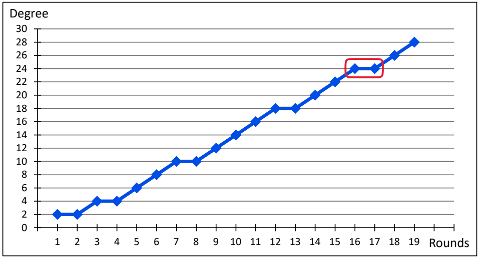

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An upper bound

Proposition

Set of exponents that might appear in the polynomial:

$$\mathcal{E}_r = \{3j \bmod (2^n - 1) \text{ where } j \preceq i,\ i \in \mathcal{E}_{r-1}\}$$

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No exponent $\equiv 5, 7 \bmod 8$ $\Rightarrow$ No exponent $2^{2k} - 1$

$$\mathcal{E}_r \subseteq \left\{ \begin{array}{cccccccc} 0 & 3 & 6 & 9 & 12 & \cancel{15} & 18 & \cancel{21} \\ 24 & 27 & 30 & 33 & 36 & \cancel{39} & 42 & \cancel{45} \\ 48 & 51 & 54 & 57 & 60 & \cancel{63} & 66 & \cancel{69} \\ & & & & & & & \\ \ldots & 3^r \} & & & & & & \end{array} \right.$$

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Example : $63 = 2^{2\times 3} - 1 \notin \mathcal{E}_4 = \{0, 3, \dots, 81\}$ $\Rightarrow B_3^4 < 6 = wt(63)$

$\forall e \in \mathcal{E}_4 \backslash \{63\}, wt(e) \leq 4$ $\Rightarrow B_3^4 \leq 4$

Bounding the degree

Theorem

After r rounds of MiMC, the algebraic degree is

$$B_3^r \leq 2 \times \lceil \lfloor \log_2(3^r) \rfloor / 2 - 1 \rceil$$

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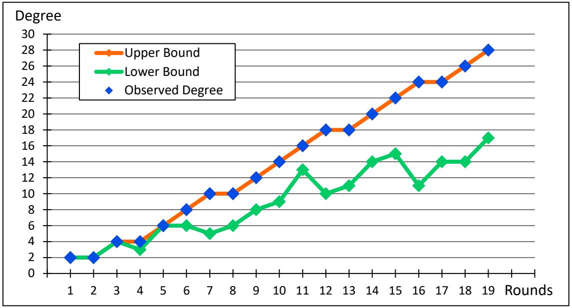
And a lower bound if $3^r < 2^n - 1$:

$$B_3^r \geq wt(3^r)$$



Exact degree

Maximum-weight exponents:

Let $k_r = \lfloor r \log_2 3 \rfloor$.

$\forall r \in \{4, \ldots, 16265\} \backslash \mathcal{F}$ with $\mathcal{F} = \{465, 571, \ldots\}$:

- if k_r is odd,

$$\omega_r = 2^{k_r} - 5 \in \mathcal{E}_r,$$

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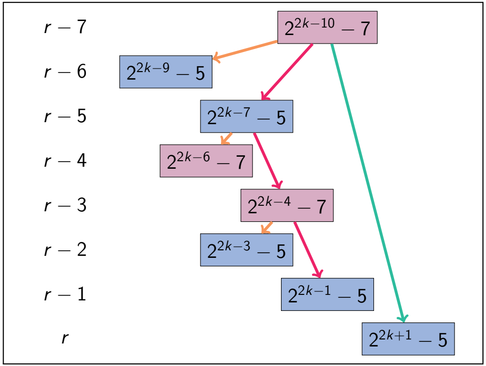

Constructing exponents.

$$\boxed{\exists\, \ell \text{ s.t. } \quad \omega_{r-\ell} \in \mathcal{E}_{r-\ell} \;\Rightarrow\; \omega_r \in \mathcal{E}_r}$$

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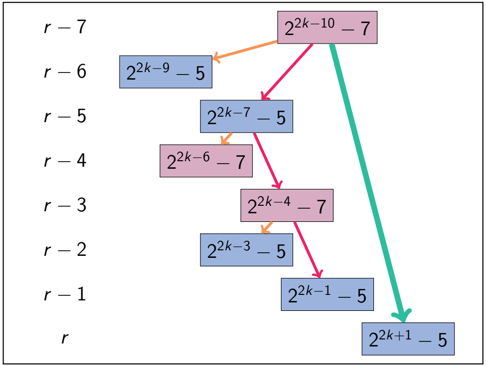

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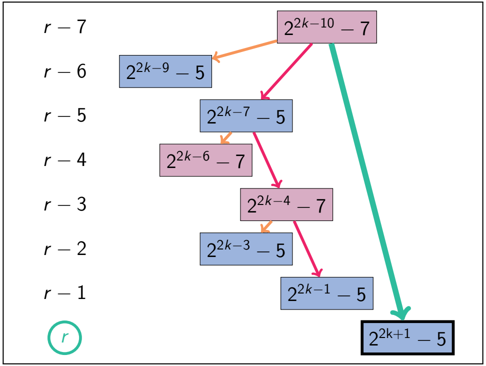

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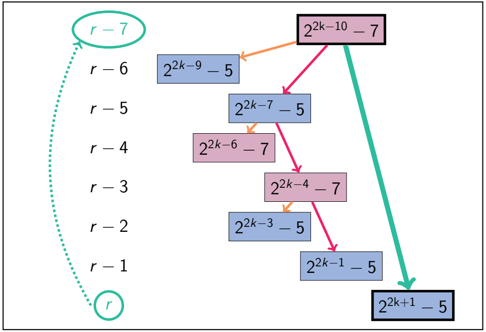

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Covered rounds

Idea of the proof:

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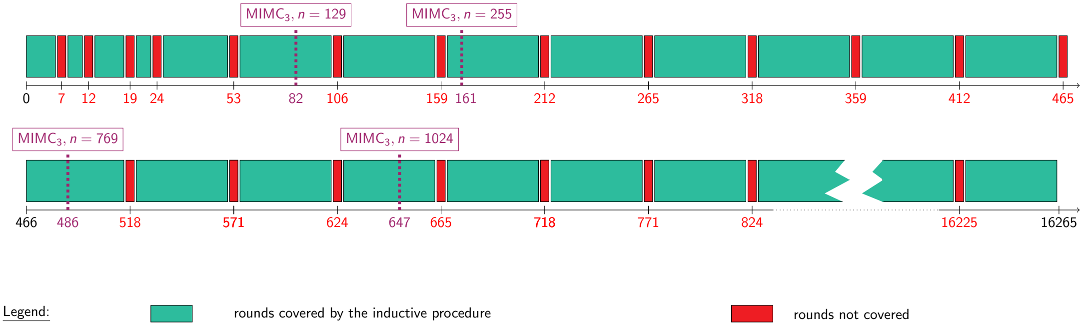

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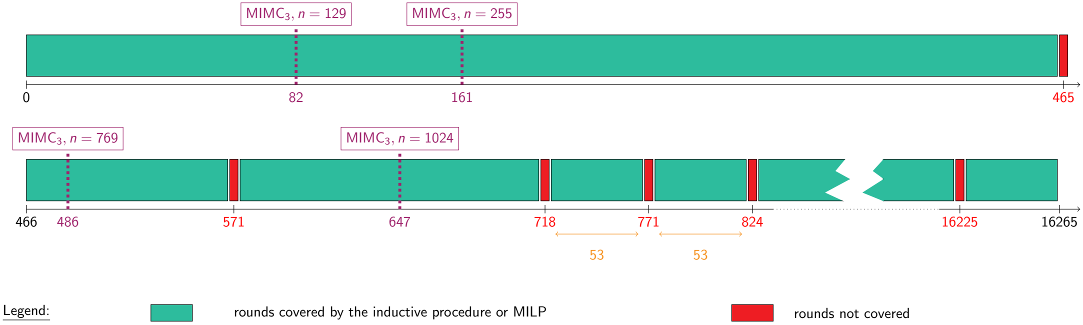

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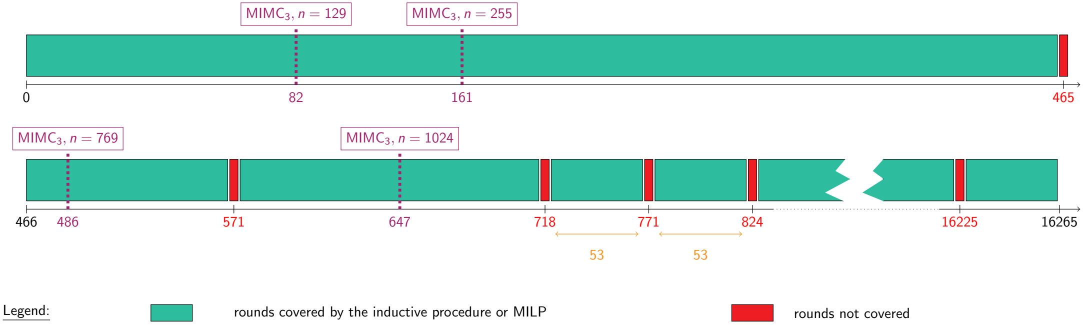

$\Rightarrow$ plateau when $k_r = \lfloor r \log_2 3 \rfloor$ is odd and $k_{r+1} = \lfloor (r+1) \log_2 3 \rfloor$ is even

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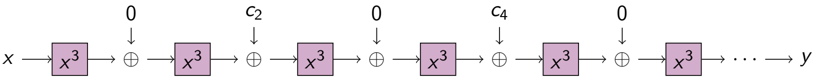
$MiMC_9$ and form of coefficients

- $MIMC_3[2r]$



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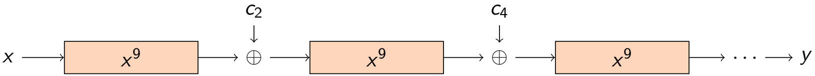

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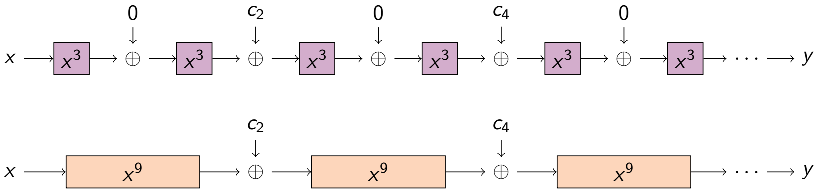

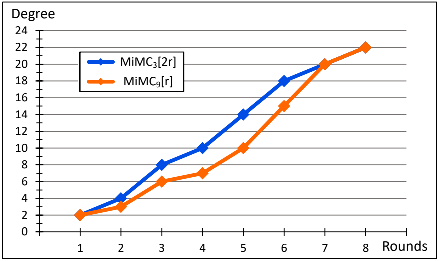

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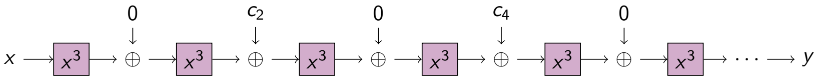
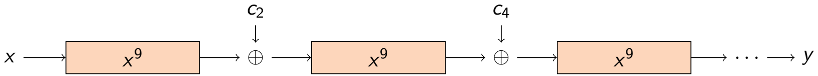
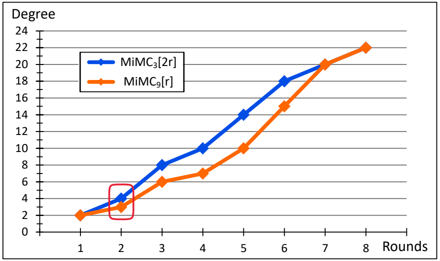


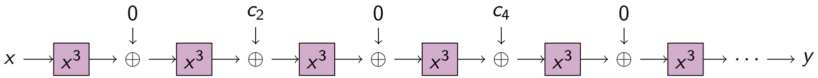
- $\mathrm{MIMC}_9[r]$





MiMC$_9$ and form of coefficients

- MIMC$_3[2r]$



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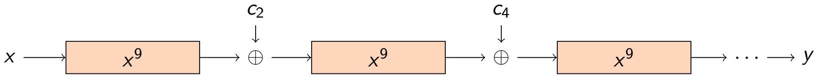

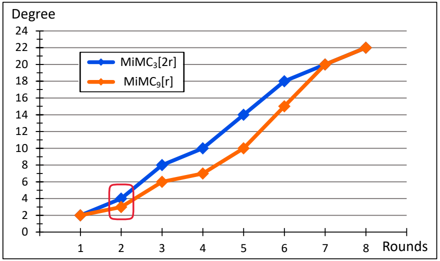

Example: coefficients of maximum weight exponent monomials at round 4

$$
\begin{array}{ll}
27 : c_1^{18} + c_3^2 & 57 : c_1^8 \\
30 : c_1^{17} & 75 : c_1^2 \\
51 : c_1^{10} & 78 : c_1 \\
54 : c_1^9 + c_3 &
\end{array}
$$

Other Quadratic functions

Proposition

Let $\mathcal{E}_r$ be the set of exponents in the univariate form of $\mathrm{MIMC}_9[r]$. Then:

$$\forall\, i \in \mathcal{E}_r,\ \ i \bmod 8 \in \{0, 1\}\ .$$

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Gold Functions: x^3, x^9, ...

$$x \longrightarrow \boxed{x^d} \longrightarrow \overset{\displaystyle c_1 \atop \displaystyle \downarrow}{\oplus} \longrightarrow \boxed{x^d} \longrightarrow \cdots \longrightarrow \overset{\displaystyle c_{r-1} \atop \displaystyle \downarrow}{\oplus} \longrightarrow \boxed{x^d} \longrightarrow y$$

Proposition

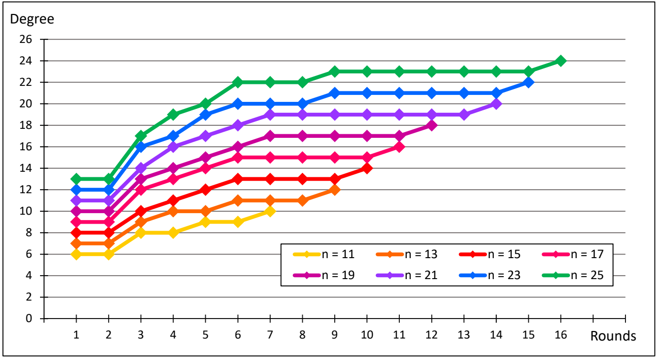
Let $\mathcal{E}_r$ be the set of exponents in the univariate form of $\mathrm{MIMC}_d[r]$, where $d = 2^j + 1$. Then:

$$\forall\, i \in \mathcal{E}_r,\ \ i \bmod 2^j \in \{0, 1\}\ .$$

Study of MiMC_3^{-1}

Inverse: $F : x \mapsto x^s, s = (2^{n+1} - 1)/3 = [101..01]_2$

$$x \longrightarrow \boxed{x^s} \longrightarrow \overset{c_{r-1}}{\overset{\downarrow}{\oplus}} \longrightarrow \boxed{x^s} \longrightarrow \cdots \longrightarrow \overset{c_1}{\overset{\downarrow}{\oplus}} \longrightarrow \boxed{x^s} \longrightarrow y$$



Some ideas studied

Plateau between rounds 1 and 2, for $s = (2^{n+1} - 1)/3 = [101..01]_2$:

- ♫ Round 1: $B_s^1 = wt(s) = (n+1)/2$
- ♫ Round 2: $B_s^2 = \max\{wt(is), \text{ for } i \preceq s\} = (n+1)/2$

Proposition

For $i \preceq s$ such that $wt(i) \geq 2$:

$$wt(is) \in \begin{cases} [wt(i) - 1, (n-1)/2] & \text{if } \; wt(i) \equiv 2 \bmod 3 \\ [wt(i), (n-1)/2] & \text{if } \; wt(i) \equiv 0 \bmod 3 \\ [wt(i), (n+1)/2] & \text{if } \; wt(i) \equiv 1 \bmod 3 \end{cases}$$

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Next rounds: another plateau at $n - 2$?

$$\boxed{r_{n-2} \geq \left\lceil \frac{1}{\log_2 3} \left(2 \left\lceil \frac{n-1}{4} \right\rceil + 1\right) \right\rceil}$$

1. Background
   - Emerging uses in symmetric cryptography
   - Definition of algebraic degree
2. On the algebraic degree of MiMC_3
   - First plateau
   - Bounding the degree
   - Exact degree
3. Other permutations
   - Quadratic functions
   - Algebraic degree of MiMC_3^{-1}
4. Integral attack
   - Secret-key 0-sum distinguisher
   - Comparison to previous work

Higher-order differential attack

Exploiting a low algebraic degree

For any affine subspace $\mathcal{V} \subset \mathbb{F}_2^n$ with $\dim \mathcal{V} \geq \deg^a(F) + 1$, we have a 0-sum distinguisher:

$$\bigoplus_{x \in \mathcal{V}} F(x) = 0.$$

Random permutation: degree $= n - 1$

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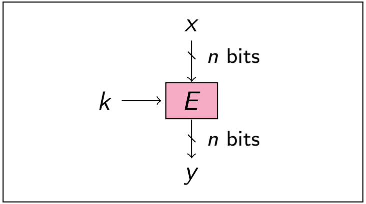
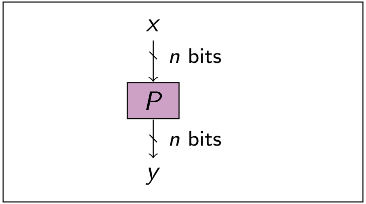


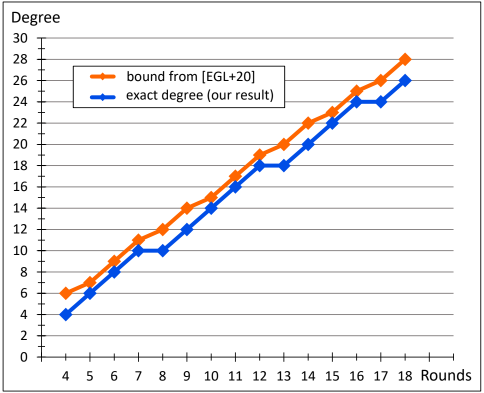
Block cipher



Random permutation

Comparison to previous work

First Bound: $\lceil r \log_2 3 \rceil \quad \Rightarrow \quad$ Exact degree: $2 \times \lceil \lfloor r \log_2 3 \rfloor / 2 - 1 \rceil$.



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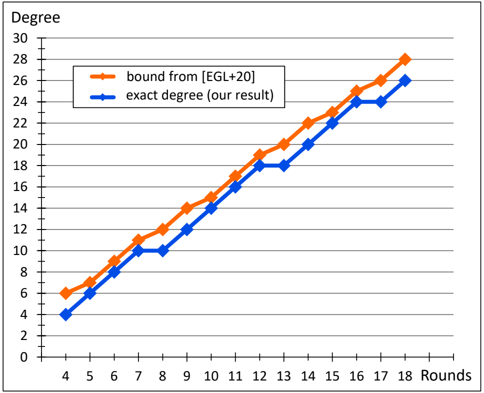

For $n = 129$, $\mathrm{MIMC}_3 = 82$ rounds

Rounds	Time	Data	Source
80/82	2^{128}XOR	2^{128}	[EGL+20]
81/82	2^{128}XOR	2^{128}	New
80/82	2^{125}XOR	2^{125}	New

Secret-key distinguishers ($n = 129$)

Conclusions

- guarantee on the algebraic degree of MIMC_3.
  - upper bound on the algebraic degree:

$$2 \times \lceil \lfloor \log_2(3^r) \rfloor / 2 - 1 \rceil \ .$$

  - bound tight, up to 16265 rounds.
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- application in music for semiconvergents of $\log_2(3)$

Thanks for your attention

Music in MIMC$_3$

- Patterns in sequence $(k_r)_{r>0}$:

$$\Rightarrow \text{denominators of semiconvergents of } \log_2(3) \simeq 1.5849625$$

$$\mathfrak{D} = \{\boxed{1}, \boxed{2}, 3, 5, \boxed{7}, \boxed{12}, 17, 29, 41, \boxed{53}, 94, 147, 200, 253, 306, \boxed{359}, \ldots\} ,$$

$$\log_2(3) \simeq \frac{a}{b} \quad \Leftrightarrow \quad 2^a \simeq 3^b$$

- **Music theory:**
  - perfect octave 2:1
  - perfect fifth 3:2

$$2^{19} \simeq 3^{12} \quad \Leftrightarrow \quad 2^7 \simeq \left(\frac{3}{2}\right)^{12} \quad \Leftrightarrow \quad \text{7 octaves } \sim \text{12 fifths}$$

Sporadic Cases

Bound on ℓ

Observation

$$\forall 1 \leq t \leq 21,\ \forall x \in \mathbb{Z}/3^t\mathbb{Z},\ \exists \varepsilon_2, \ldots, \varepsilon_{2t+2} \in \{0,1\},\ \text{s.t. } x = \sum_{j=2}^{2t+2} \varepsilon_j 4^j \bmod 3^t\ .$$

Let: $k_r = \lfloor r \log_2 3 \rfloor$, $b_r = k_r \bmod 2$ and

$$\mathcal{L}_r = \{\ell,\ 1 \leq \ell < r,\ \text{s.t. } k_{r-\ell} = k_r - k_\ell\}\ .$$

Proposition

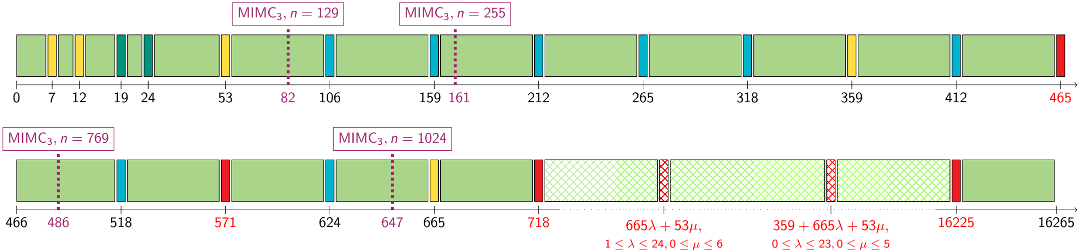
Let $r \geq 4$, and $\ell \in \mathcal{L}_r$ s.t.:

- $\ell = 1, 2$,
- $2 < \ell \leq 22$ s.t. $k_r \geq k_\ell + 3\ell + b_r + 1$, and ℓ is even, or ℓ is odd, with $b_{r-\ell} = \overline{b_r}$;
- $2 < \ell \leq 22$ is odd s.t. $k_r \geq k_\ell + 3\ell + \overline{b_r} + 5$

Then $\omega_{r-\ell} \in \mathcal{E}_{r-\ell}$ implies that $\omega_r \in \mathcal{E}_r$.

Covered Rounds

Rounds for which we are able to exhibit a maximum-weight exponent.



Legend:

Rounds for which we are able to construct an exponent.

- semiconvergents of $\log_2(3)$: MILP
- "good" ℓ
- no "good" ℓ: MILP
- no "good" ℓ ($\ell \geq 53$): MILP

Rounds likely to be covered by solving the conjecture.

- no "good" ℓ: no result with MILP

Covered Rounds

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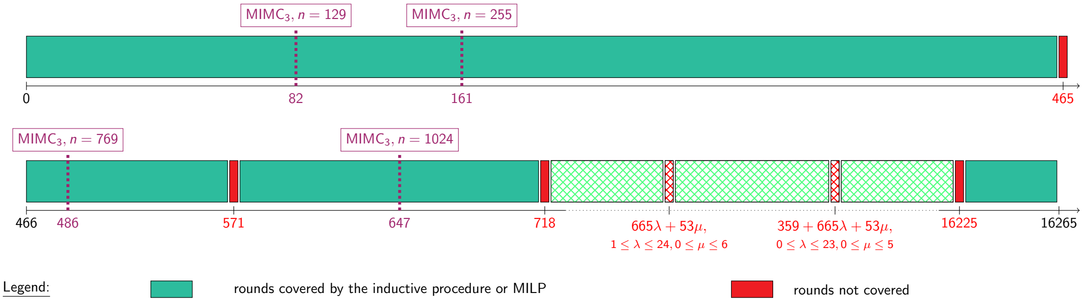


MILP Solver

Let

$$\text{Mult}_3 : \begin{cases} \mathbb{N}^{\mathbb{N}} & \to \mathbb{N}^{\mathbb{N}} \\ \{j_0, \ldots, j_{\ell-1}\} & \mapsto \{(3j_0) \bmod (2^n - 1), \ldots, (3j_{\ell-1}) \bmod (2^n - 1)\} \end{cases},$$

and

$$\text{Cover} : \begin{cases} \mathbb{N}^{\mathbb{N}} & \to \mathbb{N}^{\mathbb{N}} \\ \{j_0, \ldots, j_{\ell-1}\} & \mapsto \{k \preceq j_i, i \in \{0, \ldots, \ell - 1\}\} \end{cases}.$$

So that:

$$\mathcal{E}_r = \text{Mult}_3\big(\text{Cover}(\mathcal{E}_{r-1})\big) .$$

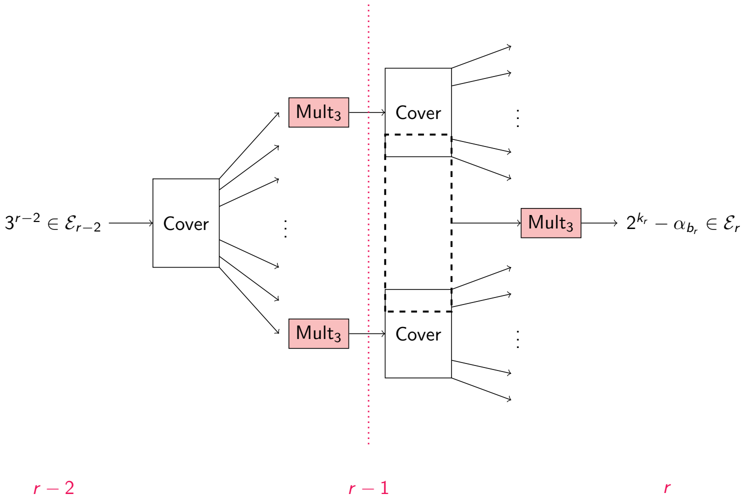
$\Rightarrow$ MILP problem solved using `PySCIPOpt`

$$\boxed{\text{existence of a solution} \quad \Leftrightarrow \quad \omega_r \in (\text{Mult}_3 \circ \text{Cover})^{\ell}(\{3^{r-\ell}\})}$$

With $\ell = 1$:

$$3^{r-1} \in \mathcal{E}_{r-1} \longrightarrow \boxed{\text{Cover}} \longrightarrow \boxed{\text{Mult}_3} \longrightarrow 2^{k_r} - \alpha_{b_r} \in \mathcal{E}_r$$

MILP Solver (2 rounds)



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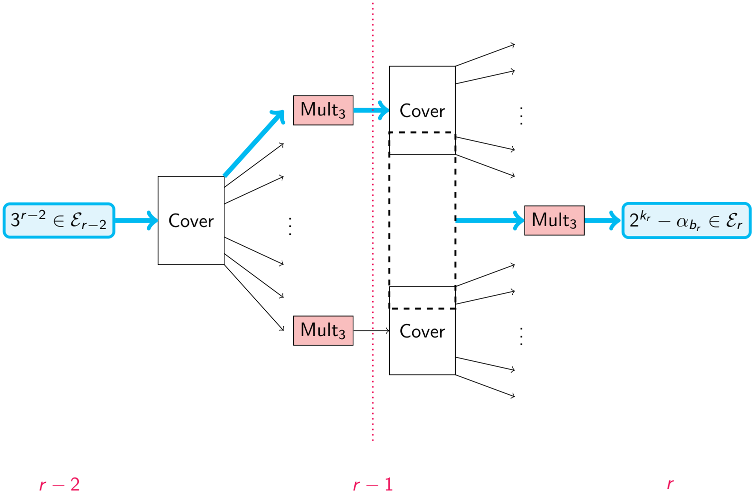

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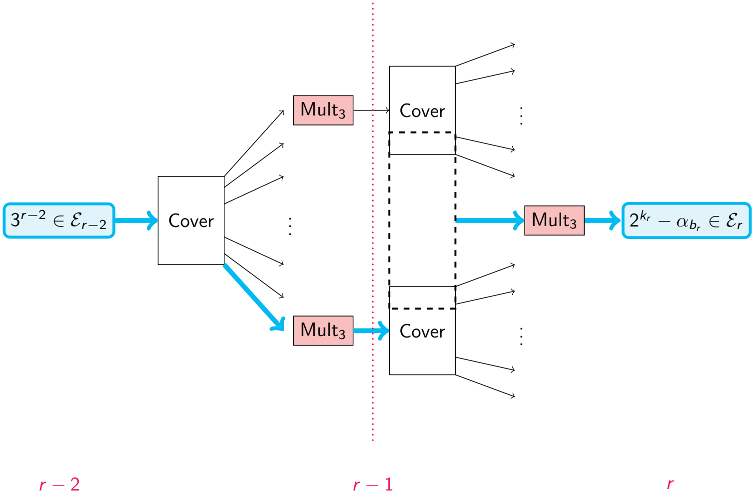
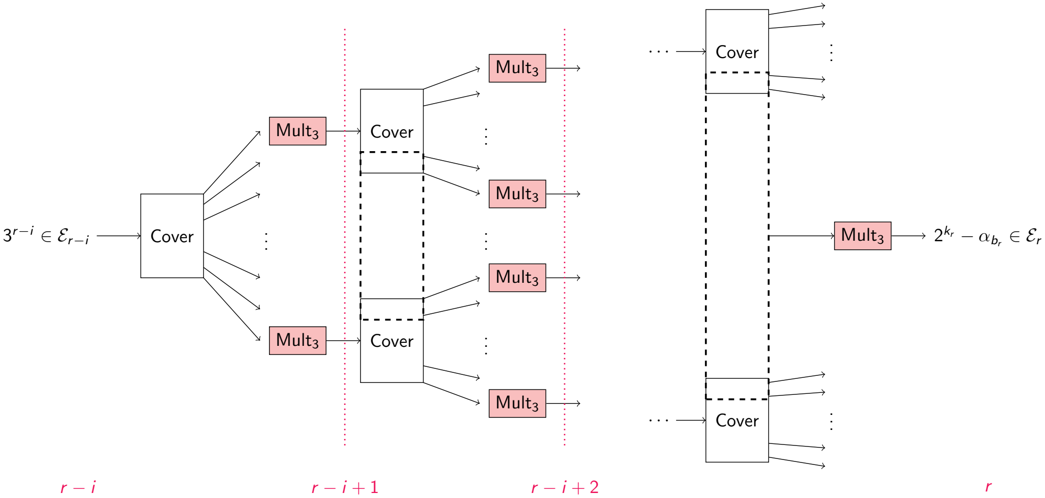


MILP Solver (i rounds)



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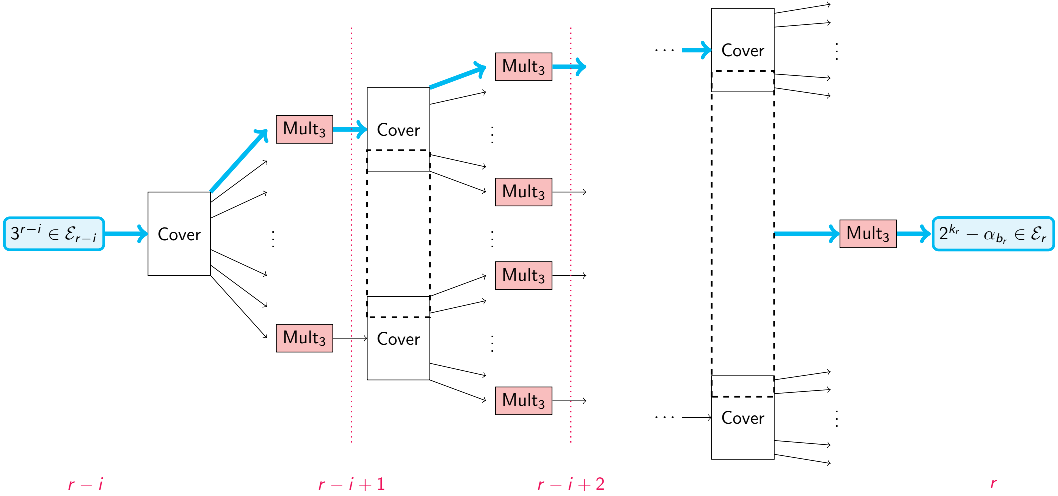

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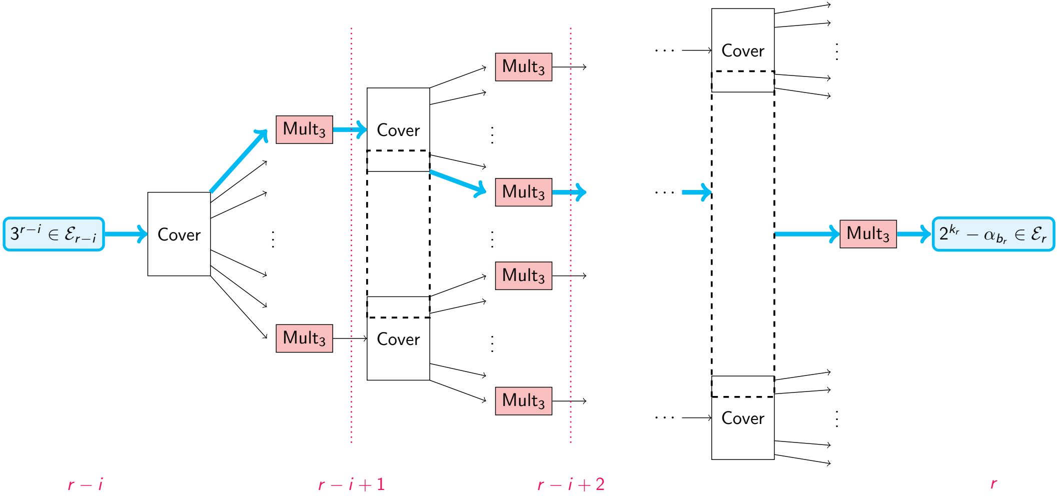

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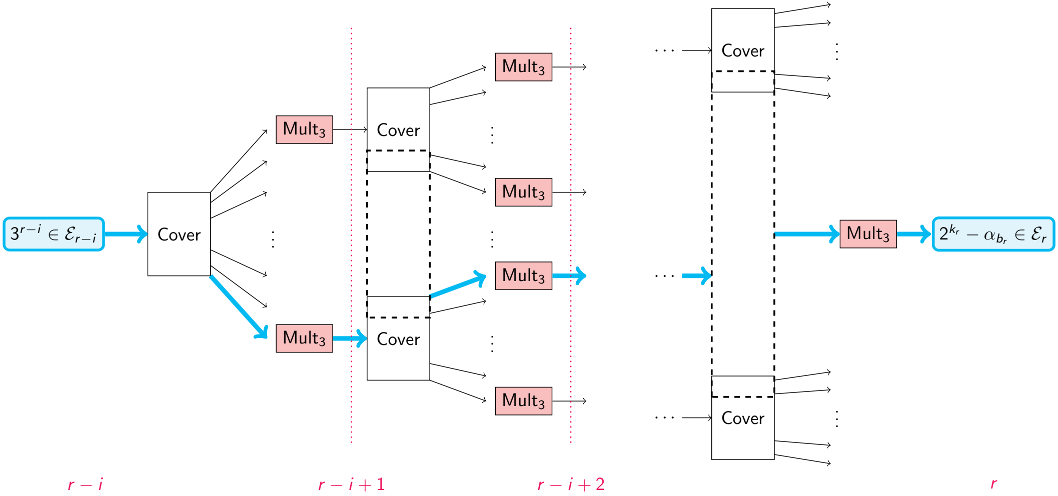

MILP Solver (i rounds)